GEOMETRICAL - RESEARCHES
ON
PERCEPTICS
IN THE
GEOMETRY OF
FOUR DIMENSIONS

By GEORGE BRANDES

LAPIS POLARIS MAGNES.

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Note; This 2nd printing has been done in black only-and lacks the red lines in the illustrations. It is suggested that the reader take a red marking device and go over the lines which have the minute dashes or cross cuts on them-to distinguish color. There is sometimes a slight depth difference to distinguish by...or notations.

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CHAPTER I

INTRODUCTION

I. FUNDAMENTAL-CONCEPTS

1. SYNESTHESIA. In 3-space Euclidean-geometry we find 3 basic-component modes of thought integrated into 1 whole: the axiomatic, analytic, and perceptic. Let us call these 3 basic components, respectively, the synthetic-component, analytic-component, and perceptic-component. Further, we abbreviate these x-components as: the S-component, A-component, and P-component. We can form a triangle to represent these combined x-components as a SAP-triangle of organization. At each corner of the triangle we have a basic-component which we call, respectively, the S-, A-, and P-component. The SAP-triangle when used gives us a great understanding of the science of geometry. The strongest component of the SAP-triangle is the P-component, the balance, is such, that an increase in any 1 component will increase the development of the other components—the ideal-situation occurs using all 3 x-components simultaneously, functionally, that is. We shall call the P-component of the SAP-triangle the 'synesthesic-unit', which, when developed along with the S- and A-units, gives us great insight and understanding of geometric-forms of thought.

In the 4-space geometries as developed to date. We have only 2 corners of the SAP-triangle extensively-developed, the axiomatic and analytic. An ultra-change occurs when you put in the P-unit; geometric-integration occurs in a 'twinkling of an eye' with unusual-insights of 4-space geometries never before suspected to have existed. The SAP-triangle 'gestalt' gives us a real-understanding of geometry per se. In my geometrical-researches for finding a general-method to solve 4-space problems quickly. I became fully-aware that a 4-space 'graphic' would require for its development a SAP-triangle approach, in such-a-way, that the 4-space perceptics would be consistent with the S- and A-components of the SAP-triangle. Assuming further, that these results would as-a-whole give enormous power in geometric-understanding and new-discoveries,

Much of the 4-space perceptics was developed with the aide of the 'inner-eye'. In visualizing 4-space geometric-figures one must focus one's attention on a new-level of mental-awareness—somewhat like the 'psychics' do in seeing the human-aura. In going from plane to solid-geometry we focus our 'attention' on a second-level of geometric-awareness, then re-focusing again to a 3rd-level for larger geometric-gestalt-units of integration for hypersolid geometric-figures.

One must go slowly in using the multiple-focusing-process of the inner-eye to visualize 4-space geometric-figures. It takes time and patience. One must go over and over again until the 'natural' geometric-gestalt occurs.

We are but babes in the world of the 4th dimension. So again I say, time and patience is required for a real-understanding of this fascinating new realm of hyperspace-geometry. I predict that the psi-discoveries resulting from applications of the 4-space geometry will be enormous—its a gateway to the stars.

THE CANONICAL-HYPERCUBE

2. GRAPHIC-CONSTRUCTION. COLOR-DIMENSION. In creating a 4-space visual-graphic that will enable us to use the SAP-triangle principle. We must begin with the simplest geometric-figure possible that can be used as a standard-form or archetype-model. The 4-space geometric-figure must have the intrinsic-properties that can incorporate directly the analytic- and synthetic-modes of 4-space geometry.

The org-theory I published in '63 gave me the conceptual-solution to this unsolved-problem. The combinatorial-integrator for the 'orgs' showed me that the simplest structure possible to form was made-up of the units-partitions of the natural-numbers. Suppose we consider the units-partitions of the natural-numbers as independent-subsets of higher order generated from the universal-set $(1,1,1,\ldots)$, then its independent-subsets we can represent as: $(1, 1^2, 1^3, \ldots)$, which represent the natural-numbers in partition-form. The combinatorial-groupings of these units-partitions gives us the combinations of higher order for the natural-numbers, i.e. the units-partition 1^4 via combinatorial-grouping gives us the 'partitions' of the natural-number 4 as follows: 21^2, 31, 2^2, and 4; it being understood that the partition-parts when summed equal the natural-number 4, the invariant-property of the partition-grouping.

Further, in group-theory, we find that the simplest existing group-structure associated with the units-partitions is the product-group of 1.

If we select the units-partitions of the natural-number 4 and relate these 4 1's to our 4-space Euclidean-geometry, we shall then have the 'master-key' that opens a door into a new 'dimension!. In the combinatorial-analysis of 4-space Euclidean-geometry we associate the unit-parts of the units-partition of the natural-number 4 with the 4-space mutually perpendicular coordinate-axes. In 4-space we have a set of 4 mutually perpendicular-lines that meet at a point called the origin. These 4 mutually perpendicular-lines are all 90° apart from one another, which can be grouped together in different ways, like the partitions of the number 4. This amount of information gives us the 'synthetic-number-partitions' that relate our canonical-hypercube to analytic-geometry and the calculus.

Further, we need a few bits more of information to graphically-construct the 4-space canonical-hypercube. This additional info we find in the fully developed 4-space synthetic-geometry, from which, we make use of the important concepts of 'boundary' and 'interior' of a geometric-figure. One more step completes the synesthesic-process for visualizing the canonical-hypercube. Since we will be 'visualizing' in 4-space perceptics, our 4-space graphics will be much more involved than the ordinary 3-space solid-geometry. Therefore, in the 4-space graphics another change is required. We make use of 2 colors to distinguish our ordinary 3-space 'cell' from the other regions in hyperspace, i.e. a black-cube for our ordinary-space and 7 red-cubes for the remaining portions of our canonical-hypercube in hyperspace.

In setting a graphic mock-up of the canonical-hypercube we have yet to consider 'double-oblique-projections' in the hyperspace-graphics. The x- and z-axes lie in the plane of the paper at true-right-angles to each other. The y- and w-axes are represented as being at quasi-right-angles to each other in the plane of the paper, as well as, 1 of each of these 2 with 1 of each of the former 2. The 2 quasi-perpendicular-axes have been chosen, in such-a-way, that in the plane of the paper, the graphic-form of the canonical-hypercube will have 8 of its vertices in the interior of an octagon—the 3-space analogue to this being the graphic-cube having 2 of its vertices lying in the interior of a hexagon. The 'octagon' is slightly doubly-distorted due to the double-oblique-projection in the 4-space graphic of the canonical-hypercube. From graphic-experimentation I have come across the graphic-design that makes the 4-space visualization-process occur with ease without undue mental-stress; in special-cases, the standard-design of the graphic-coordinates can be modified to stress better pictorial-relationships of a few of the 4-space geometry theorems.

At the end of this chapter on the last-page is the graphic-form for the canonical-hypercube. On plate-I near the lower-right-hand-corner is the standard-code for the graphic-construction of the canonical-hypercube. To 'save time', the student should make a duplicate-copy of the canonical-hypercube on plate-I, and then continue with the study of this chapter by referring to the duplicate-copy.

We shall now explain the standard-code as it relates to the graphic-construction of the canonical-hypercube.

In the plane of the paper we have 2 half-lines OX and OZ intersecting in a right-angle at the point O. On each of these ½-lines we establish a numerical-scale of positive-real-numbers. The vertical ½-line OZ is labelled the positive-portion of the z-axis, the horizontal ½-line OX is labelled the positive-portion of the x-axis. In the plane of the paper the numerical-scales chosen for the x- and z-axes are the same. The right-angle at O of the 2 positive-portions of the x- and z-axes is a true-right-angle which can be measured directly—the term 'true' is used in the sense that the right-angle at O of XOZ can be measured directly in the graphic, and is a true-right-angle having 0-distortion in the plane of the paper, that is, only in the graphic-viewpoint.

On the oblique ½-line OY as constructed in the plane of the paper, we establish a numerical-scale of positive-real-numbers. The oblique ½-line OY is labelled the positive-portion of the y-axis. In the oblique-direction of the positive-portion of the y-axis, we have a moderate-foreshortening of its length to adjust for the moderately-oblique parallel-sweep of lines that 'appear' longer in length when nearly perpendicular to the observer's 'line of sight'.

Therefore the numerical-scale attached to the positive-portion of the y-axis is foreshortened by '$\frac{1}{2}$-inch'. In the plane of the paper we also make another adjustment for the perpendicular $\frac{1}{2}$-line OY, in such-a-way, that the $\frac{1}{2}$-lines OZ and OY intersect at O in an obtuse-angle, i.e. the obtuse-angle ZOY.

On the 2nd oblique $\frac{1}{2}$-line OW as constructed in the plane of the paper, we establish a numerical-scale of positive-real-numbers. The 2nd oblique $\frac{1}{2}$-line OW is labelled the positive-portion of the w-axis. In the oblique-direction of the positive-portion of the w-axis, we have a smaller-foreshortening of its length to adjust for the smaller-oblique parallel-sweep of lines that 'appear' somewhat longer in length when less perpendicular to the observer's 'line of sight'. Therefore the numerical-scale attached to the positive-portion of the w-axis is foreshortened by '$\frac{1}{4}$-inch'. In the plane of the paper we also make another adjustment for the perpendicular $\frac{1}{2}$-line OW, in such-a-way, that the $\frac{1}{2}$-lines OZ and OW intersect at O in an acute-angle, i.e. the acute-angle ZOW.

The student should be aware that we have used only the positive-portions of the 4-space Cartesian-coordinate-system in the graphic-construction of the canonical-hypercube—extensions of the 4-space Cartesian-coordinates in the negative-directions of the positive-coordinates from the point O are easily made.

We shall call the 4-space graphic-projection of the canonical-hypercube, 'double-oblique-projection'.

In completing the graphic-construction of the canonical-hypercube for hidden-views, the graphic-process is similar to the 3-space solid-geometry, but with some important differences—the development of the 4-space geometric-figures with hidden-views will be taken up later in another portion of this chapter, and in the chapters that follow.

3. CONFORMABLE-ORIENTATION-SENSE. INVARIANCE. Given 4 mutually perpendicular-hyperplanes of 'cubes' intersecting at the point O, such that any 2 hyperplanes of the cubes intersect in the plane of a square, that is, any 2 adjacent-cubes at the point O will intersect in a common-face belonging to both of the adjacent-cubes.

Since there are degrees of perpendicularity as well at the point O, we can 'define' the dimension of a geometric-form by associating it with the maximum-degree of 'perpendicularity' that it can contain at a given point—the differential-geometry of hypersurfaces would be an example of this for local-neighborhoods of hyperspace having dimensional-invariance. The perpendicularity-principle is the root-assumption for all our metric-geometries—Pythagorean-theorem would be such an example.

We call the canonical-hypercube a geometric-form having perpendicularity of the 4th degree, thereby, giving us the 4-space Euclidean-geometry and its derivatives, such as the Non-Euclidean-geometries—the Euclidean, we call static-geometry, the Non-Euclidean, dynamic-geometry.

Take the intersection of the hyperplane of the black-cube with 1 of the 3 intersecting hyperplanes of the red-cubes that intersect in the perpendicular-line of the segment OO', say the hyperplane of the red-cube OBHC—O'B'H'C'. Then the hyperplane of the black-cube intersects the hyperplane of the red-cube OBHC—O'B'H'C' in the plane of the black-square OBHC. Now in the hyperplane of the black-cube the plane of the black-square OBHC lies obliquely to our left. The side of the plane of the black-square OBHC that we see lies towards the 3-space interior of the black-cube. In the hyperplane of the red-cube OBHC—O'B'H'C', the side of the plane of the same black-square OBHC that we see lies towards the outside of the red-hyperplane-cube OBHC—O'B'H'C', i.e. we still see the same-side of the plane of the black-square OBHC. In the hyperplane of the red-cube OBHC—O'B'H'C' we must then have the plane of the red-square OO'C'C lie obliquely towards our right. For suppose the plane of the red-square OO'C'C were to lie obliquely towards our left, i.e. somewhat upwards and passing by us on the left somewhat distant. Then we would see the other-side of the plane of the black-square OBHC, with this other-side pointing in the direction towards the interior of the red-cube OBHC—O'B'H'C'. But this 2nd orientation-sense of the plane of the red-square OO'C'C belonging to the hyperplane of the red-cube OBHC—O'B'H'C' would alter the orientation-sense of the hyperplane of the black-cube, for now we would see the opposite-side of the plane of the black-square OBHC slant-obliquely towards us on our right, thus changing the front and backviews of the hyperplane of the black-cube; the front-view BHFE of the black-cube would now become the back-view OCGA, and the back-view OCGA would become the front-view BHFE.

This 'domino-effect' spreads to the hyperplanes of the other 2 red-cubes OCGA-O'C'G'A' and OAEB-O'A'E'B'. Since we are given that the hyperplane of the red-cube OCGA-O'C'G'A' intersects the hyperplane of the red-cube OBHC-O'B'H'C' in the plane of the red-square OO'C'C which contains the line of the segment OO'. Then in the hyperplane of the red-cube OCGA-O'C'G'A' which also intersects the hyperplane of the black-cube in the plane of the black-square OCGA, we would then have the plane of the red-square OO'C'C lie towards us on our left, making the front- and back-views of the red-cube OCGA-O'C'G'A' interchange positions, thus altering the orientation-sense of the hyperplane of this red-cube. Similar results occur as well with the hyperplane of the red-cube OAEB-O'A'E'B' intersecting the hyperplane of the red-cube OBHC-O'B'H'C' in the plane of the red-square BB'O'O. Therefore the plane of the red-square OO'C'C containing the positive-portion of the w-axis lies towards us on our left.

The conformable-orientation-sense of the hyperplane of 1 cube <u>induces</u> the <u>same</u>-conformable-orientation-sense for all the adjacent-hyperplanes of the cubes constructed on the faces of the cubes lying in the different hyperplanes that have been made conformable to the hyperplane of the original cube.

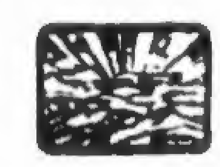
The <u>invariance</u> of the conformable-orientation-sense for a network of hyperplanes of the cubes can be put into a theorem as follows:

Given the conformable-orientation-sense for the hyperplane of 1 cube induces a conformable-orientation-sense for all the other hyperplanes of the cubes constructed about the hyperplane of the given cube.

The principle of the conformability of orientation-sense for hyperplanes of the cubes makes it possible for us to have hidden-views in the double-oblique-projection of the canonical-hypercube as well as eliminating optical-illusion effects.

Something like the conformable-orientation-sense would occur if we were to 'transmit' the decomposed 'electromagnetic-units' of the molecular-structure of 'matter', that is, for a certain unknown-wave-band in the matter-spectrum; the stamped-letters on an ash-tray would read-out <u>backwards</u> if we disregarded the conformable-orientation-sense of molecular-networks.

Ia. 4-SPACE INSIGHTS



4. THE MATRIX-GRID OF POINTS. In order to view a 'true' 4-space geometric-figure having the <u>minimum</u> of space-distortion we must be able to see from <u>outside</u> of the 3-space hypersurface, which with our limited <u>outer-eye</u> we can not do because its structural-boundaries are 2-dimensional. But there is a partial-way out of this dilemma. We can use our inner-eye as a lens-type of focusing-apparatus from which to view the graphic-form of the canonical-hypercube. We can consider this possibility: suppose we let the <u>dimension</u> of <u>color</u> represent the points in hyperspace <u>outside</u> of our 3-space 'cell' <u>together</u> with the double-oblique-projection graphic-principle. Then, we can, within our capabilities, see with the aide of the inner-eye through 2nd-level focusing, a very close-approximation of the 'ideal-hypercube'. Let us see if our inner-logic is valid?

If our 'viewpoint' was that of the flatlander, i.e. <u>in</u> 2-space, then we would have <u>insufficient-space</u> to form a 2-space <u>matrix-grid</u> <u>of</u> <u>points</u>. In flatland we would have a <u>true</u>-line <u>on</u> which to represent 1 of the edges of a square as a <u>true</u>-edge, whereas, the oblique-line representation <u>in</u> the flatlander's 2-space does <u>NOT-exist</u>; the reason, for this non-existence of an oblique-line <u>in</u> the flatlander's 1-line-graphic representation is obvious: only 1 line can be drawn perpendicular to a given line through a given point of the given line, but <u>in</u> the flatlander's 1-line-graphic we have <u>no</u> perpendicular-lines that can lie on the <u>same</u>-line. However, the flatlander does have a partial-way-out of this 1-space <u>constraint</u>. He can represent 2-space 'perceptics' <u>on</u> his 1-line-graphic as <u>degrees</u> of <u>foreshortening</u> of line-segments. In flatland then, we have <u>actually</u>, on the 1-line-graphic, <u>relations</u> amongst sets of points <u>classified</u> into various size segment-lengths that on the 1-line-graphic <u>overlap</u>; overlapping-points 'coded' in such-a-way as to give the flatlander his 'illusion' of <u>depth</u> as <u>seen</u> on the 1-line-graphic—2 sides of the square will have 1 true- and 1 foreshortened-edge, i.e. the 3 points forming the segments which represent 2 of the squares boundaries, and the 4th point which represents a <u>hidden-point</u> of 1 corner of the square will overlap on the

1-line graphic and coincide with 1 of the points of the interior of the segment representing the true-edge; we can code this ovelapped-point to represent a hidden-point as well as a visible-point on the 1-line-graphic. Circles will 'appear' to the flatlander as always having the same-length projections on the 1-line-graphic, other geometric-forms could be coded by possible color-shadings and color-intensities using a complex-color-scheme.

In the 3-space graphic-representation of a cube we have 2 true-lines and 1 oblique-line. The matrix-grid of points 'appears' for the first-time in 3-space as we view the 2-space matrix-grid of points outside of the plane of the paper. The remarkable feature of the matrix-grid is that we can represent hidden-views in our graphic-forms of geometric-figures as well as the conformable-orientation-sense.

In viewing a geometric-figure as-a-whole requires that we have a matrix-grid of points that can be formed into 'classes' of points and represented graphically as geometric-figures made-up of combinations of these classes of points, in other words, different geometric-gestalt-formations, by this we mean, different orders of geometric-gestalt-units integrated as-a-whole to form the 3-space geometric-figures. Hidden-views in the graphic-forms can only occur if we have a matrix-grid of points, thus, allowing us to use certain sets of free dummy-points occupying other portions of the 2-space matrix-grid.

Our limitations in the 4-space perceptics pertains to the 2-space matrix-grid of points in the plane of the paper having certain space-constraints, i.e. we can graphically represent at any point 0 on the plane of the paper, 2 lines that intersect in a right-angle, being such, that any other line through the point 0 of this intersection must be an oblique-line. These results follow from the intrinsic-geometry on the plane of the paper, i.e. that in the 2-space Euclidean-geometry only 1 line can be drawn perpendicular to a given line at a given point of the given line—as on the flat-surface of this paper.

In the 4-space perceptics, the hyperspacelander will make use of a 3-space matrix-grid of points, and viewed by him from outside of the 3-space hypersurface, thus, enabling him to use only 1 oblique-line and 3 true-lines in his graphic-forms. In the 4-space perceptics we make use of a 2-space matrix-grid of points, and viewed by us from 'outside' of the 2-space surface, thus enabling us to use 2-oblique-lines and 2 true-lines in our graphic-forms.

In the graphic-construction of the ideal-cube which 'approximates' well with the 3-space perceptics of the observer, we then have the following situation: all 6 faces of the cube are represented as 'parallelograms' in the 3-space graphics, i.e. in the plane of the paper 2 faces of the cube will be represented as having its boundaries as true-squares and the remaining 4 faces of the cube will be represented as having its boundaries as parallelograms—this will then be our ideal-visualization of 3-space geometric-forms in real 3-space; further, we have that any geometric-figure such as a square or circle when not perpendicular to the observer's 'line of sight' will appear as either a parallelogram for a square or an ellipse for a circle, the limit being that both of these geometric-figures vanish when the plane on which the geometric-figures are drawn lies 'parallel' to the 'line of sight' of the observer at eye-level, that is, with 1 dimension vanishing, we shall then see the edge-views of squares, circles, and other such edge-views of the 2-space geometric-figures.

Now the hyperspacelander has 1 advantage in his 4-space perceptics that we do not have. He is able to form a 3-space matrix-grid of points making it possible for him to have 3 true-lines in his 4-space graphic-representation of geometric-figures upon a 3-space hypersurface, i.e. being outside of our 3-space cell he can actually see all-at-once the 3 true-perpendicular-lines intersecting at a point in his 3-space hypersurface on which he makes a graphic-construction of his ideal-hypercube as seen by him in his 4-space exact-perceptics. Further, the hyperspacelander being actually in 4-space with 4-space 'perceptors' makes it possible for him to use but 1 oblique-line for 1 of his 4-space axes, i.e. the y-axis being the oblique-line in his 4-space graphic-forms, further, this oblique-line of the y-axis is actually graphically-constructed on his 3-space hypersurface.

We in 3-space can not do this amazing feat of the 4-space single-oblique-projection of 4-space geometric-figures. The best that we can do is vis-a-vis double-oblique-projection of the 4-space graphic-forms of geometric-figures.

Let us compare the viewpoint differences between the hyperspacelander's single-

oblique-projection of the canonical-hypercube and our double-oblique-projection of it. We shall assume that the hyperspacelander's oblique-line OY lying <u>in</u> his 3-space hypersurface on which his 4-space graphic-drawings are made corresponds to our oblique-line OY lying <u>in</u> our 2-space surface on which our 4-space graphic-drawings are made, such as the canonical-hypercube given on plate-I.

The hyperspacelander would <u>view</u> the 2 red-cubes OCGA-O'C'G'A' and BHFE-B'H'F'E' as <u>true-cubes</u> <u>without</u> <u>any</u> <u>space-distortions</u> whatsoever, whereas, we have space-distortions created by using the 1 oblique-line of the w-axis. The hyperspacelander's 'perceptics' of the 2 true-cubes as seen by him from his 4-space viewpoint <u>outside</u> of the 3-space hypersurface, would be to us, as our perceptics are of the 2 true-squares OCGA and BHFE as seen by us from our 3-space viewpoint <u>outside</u> of the 2-space surface. In the hyper-space of the hyperspacelander, the 2 true-cubes of the ideal-hypercube will be seen by him as having true-squares on all 6 of the faces for each of these true-cubes, whereas, <u>we</u> <u>see</u> the squares on 4 of the faces of either 1 of these true-cubes <u>'appearing'</u> as 'parallelograms' in our graphic-drawings. Further, 6 of the hyperspacelander's 'cubes' will 'appear' somewhat-similar to our black-cube, NOT the same-as, but much like our 3-space black-cube appears to us in the single-oblique-projection. Since the hyperspacelander has 3 true-lines, or equivalently, 3 coordinate-axes as true-perpendicular-axes representation, then with only a single oblique-axis, he will have 6 cubes <u>similar-in-appearance</u> to our black-cube as he <u>views</u> these 'cubes' from his <u>viewpoint</u> in hyperspace <u>outside</u> of the 3-space hypersurface; he sees these 6 cubes having the 'appearance' of <u>parallelopipeds</u>. We see something 'similar', but also <u>different</u>: 6 of the cubes of the hypercube will be seen by us also as parallelopipeds <u>as-we-view</u> the hypercube of our double-oblique-projection <u>from</u> <u>outside</u> of the 2-space surface of the paper. We will see the appearance of the 2 red-cubes OCGA-O'C'G'A' and BHFE-B'H'F'E' as parallelopipeds <u>somewhat-like</u> the appearance of our black-cube; the 2 red-cubes OBHC-O'B'H'C' and AEFG-A'E'F'G' will have the appearance of parallelopipeds <u>somewhat-more-distorted</u> than the appearance of our black-cube; the remaining 2 red-cubes OAEB-O'A'E'B' and CGFH-C'G'F'H' will have the appearance of parallelopipeds also somewhat-more-distorted due to the double-oblique-lines of the double-oblique-projection.

In summary then, our hyperspacelander has 1 oblique-line and 3 true-lines in his 4-space graphic-construction of the ideal-hypercube <u>on</u> his 3-space hypersurface; whereas, we have 2 oblique-lines and 2 true-lines in our graphic-construction of the quasi-ideal-hypercube <u>on</u> our 2-space surface, such as, the plane of the paper.

This is still a very-close-approximation for us in using the double-oblique-projection of the quasi-ideal-hypercube in studying 4-space geometric-forms, though, with somewhat-distorted inner-space-perception.

5. SCALE-DISTORTION-TABLES. Let us compare the <u>scale-distortions</u> between the ideal-hypercube graphic-representation and our quasi-ideal-hypercube graphic-representation:

Scale-Distortion Effects

Single-Oblique-Line	Double-Oblique-Lines
2 true-cubes—each having 6 true-squares;	4 partial-true-cubes—each having 2 true-squares;
6 3-space 'cubes' represented as parallelopipeds and somewhat-similar to our black-cube;	4 non-true-cubes with 0 true-squares for each of the faces of these cubes, all represented as parallelopipeds;
4 hidden-views in the graphic-form;	4 hidden-views in the graphic-form;
x sectional-views can be shown;	x sectional-views can be shown;
...	...
1 <u>scale-distortion-factor</u>.	2 <u>scale-distortion-factor</u>.

On the next-page we give 2 distortion-tables and the resulting 4-space <u>graphic-distortion</u> as relating to our quasi-ideal-hypercube.

Assign the numerical-value of 1 to a coordinate-axis, if the 'line' on which the coordinate-axis lies is a true-line, and assign the numerical-value of 0, if the 'line' on which the coordinate-axis lies is an oblique-line. It should be understood that these numerical-values are scale-values assigned to the graphic-coordinates of the canonical-hypercube.

DOP

x	y	z	1	0	1	2
x	y	w	1	0	0	1
y	z	w	0	1	0	1
z	w	x	1	0	1	2

Table-I

SOP

x	y	z	1	0	1	2
x	y	w	1	0	1	2
y	z	w	0	1	1	2
z	w	x	1	1	1	3

Table-II

The scale-distortion-index is defined as the ratio between the sum of the 0's in Table-I for the single-oblique-projection to the sum of the 0's in Table-II for the double-oblique-projection, i.e.

$$\text{Index of scale-distortion} = \frac{\text{the sum of the 0's in Table-I}}{\text{the sum of the 0's in Table-II}} = 2.$$

In the scale-distortions shown in Table-II, i.e. the 0's present in this table, shows us that in the 4-space geometric-figures represented by the 3-space hypersurface-graphics, we still have some scale-distortions, so we 'define' the scale-distortion-factor for the hypersurface-graphics as the sum of these 0's divided by the same-sum of 0's, which gives us the scale-distortion-factor of 1.

In the last-row of Table-II, we have no 0's, and the sum of the 1's is 3, so that the scale-distortion-factor in this hyperplane of the cube will be 0.

From the above comparison-tests for the scale-distortion 'effects', we saw that the scale-distortion-factor was the principle item in our list that was essentially different from the viewpoints of the hyperspacelander's graphic-forms of geometric-figures. Therefore our original assumptions are valid.

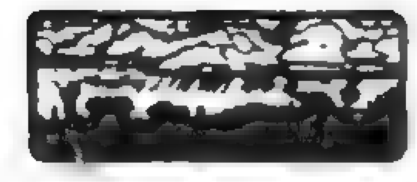
The student of this treatise can supply many of his own variations of some of the developments of the above hyperspace-perceptics, i.e. letting 1 of the coordinate-axes vanish, and then viewing the hyperplane of the remaining cube, and so fourth.

In the application of the hyperspace-perceptics to 4-space hypersolid-analytic-geometry, allowances must be made for the scale-distortions in the metric-geometry when making graphic-forms of the double-oblique-projection.

We shall abbreviate the 'single-oblique-projection' as SOP, and the 'double-oblique-projection' as DOP—this will save us a great deal of time in the repetition of these phrases.

Before leaving this section, let us briefly discuss some of the additional geometric-properties between the SOP- and DOP-graphics as shown in the scale-distortion-tables. Close-observation between relationships in both of the above tables shows us that in a true-cube the distortion-factor vanishes in real 4-space, i.e. 1 and only 1 of the hyperplanes of the cubes at the point O in the SOP-graphics will be in non-oblique-projection without any space-distortion, this corresponds to the red-cube OCGA-O'C'G'A' in the DOP-graphics. This is truely a remarkable principle relating the space-distortion of objects seen in the same-space of the observer. In the DOP- and SOP-tables we seem to have a scale-distortion-identity in the xyz-hyperplane, i.e. the scale-distortions are equal, but we know that in the 4-space SOP-graphics, that the cube represented in this hyperplane will be perceived differently when viewed in the DOP-graphics of the hypercube—similar, but not identical, as the oblique-direction of the y-axis in the SOP-graphics will lie somewhat-differently due to the 3-space hypersurface on which the 4-space graphic-forms of geometric-figures are made. The 'narrative' form used for the remaining 2 sections of this chapter on further developments of the hyperspace-perceptics should be of interest to the general reader of this treatise. The last-section of this chapter should prove interesting to the 'neophyte-geometer'.

Iaa. THE FLATLANDER'S PROJECTION INTO THE 3rd DIMENSION



6. OBLIQUE-SYMMETRY. HIDDEN-VIEWS. Suppose we in 3-space considered the flatlander's geometric-interpretation of the black-square OAEB. We would rightly 'infer' from the flatlander's viewpoint that the edges AE and EB of the black-square OAEB would be visible-views, whereas, the edges OA and OB would be invisible-views; the points A and B of the edges OA and OB of the same black-square would likewise be visible to the flatlander. Hidden-views of the edges OA and OB of the same black-square he can not represent on his 1-line-graphic due to the insufficiency of 'space-points' in his flatland. The flatlander knows from his 2-space experience that on his 1-line-graphic representation of the black-square OAEB, the point O together with the interiors of the segments OA and OB are invisible-views. The black-square OAEB as seen from the flatlander's viewpoint in flatland will be viewed by him in the following way: $\frac{1}{2}$-part of the black-square OAEB being $\frac{1}{2}$-visible and $\frac{1}{2}$-part of the same black-square being $\frac{1}{2}$-invisible.

We shall call the 3-space 'oblique-projection' of a geometric-figure having $\frac{1}{2}$-part visible and $\frac{1}{2}$-part invisible as being in standard-oblique-form, i.e. in the single-oblique-projection as we view the black-cube from our position in 3-space, that is, like we see the black-cube in the figure of the canonical-hypercube given on plate-I. This remarkable property of space-viewpoints extends to all the higher-spaces. We define the space-viewpoint for standard-oblique-forms as oblique-symmetry, such that, $\frac{1}{2}$-part of the boundaries of a geometric-figure are visible and the other $\frac{1}{2}$-part of the boundaries are invisible.

Let us project our flatlander into the spacelander's dimension giving him the latent-powers of depth-perception together with a 3-space energy-form enabling him to 'manifest' in physical-form 3-space symbol-objects, and further, enabling him to generate mental-forms of these symbol-objects in his 3-space graphics. Further, the transformed-flatlander having the abilities of 3-space perceptics can 'create' on a flat-sheet of paper his geometric-representations of 3-space solids. We assume that the transformed-flatlander's synesthesic-reasoning abilities have been extended in order to study and develop the geometry of 3-space ideal-solids. He immediately relates many of his 2-space 'experiences' as having parallels in the spacelander's dimension—refinements of his synesthesic-sense will occur later as he develops his stereometric-perception.

Suppose we let our transformed-flatlander lie in the hyperplane of the black-cube and at a little distance from above the point F—equivalently as being in our space-position as we view the hyperplane of the black-cube in the canonical-hypercube as shown on plate-I. A spectacular-change occurs in the transformed-flatlander's new founded synesthesic-sense. Our transformed-flatlander sees the point F as he did with a similar point in the 2-space black-square OAEB. For now he sees that what before was a mere point E in the 2-space black-square OAEB, has now traced-out a line-segment EF in 3-space. The edges EA and EB of the black-square OAEB now become part of the boundaries of the black-squares FEAG and FEBH. Our transformed-flatlander discovers another new remarkable relationship: a 3rd plane parallel to his 2-space plane of flatland passes through the point F and intersecting the planes of the other 2 black-squares FEAG and FEBH in the edges FG and FH. The 3 black-squares FEAG, FEBH, and FHCG are mutually-perpendicular and intersect in the point F; the plane parallel to the plane of the black-square OAEB and passing through the point F contains the black-square FHCG. He sees for the first-time his 2-space interior, i.e. what was the interior of his 2-space black-square OAEB now becomes the boundary of a 'higher-figure' in 3-space called a 'solid'. He now sees for the first-time both of the 2-space boundaries of the black-squares FEAG and FEBH as well as their interiors. The 3rd plane of the black-square FHCG intersecting at the point F the other 2 planes of the black-squares at this point, is also visible to him as well as the interior of the black-square FHCG.

The transformed-flatlander discovers one of the most unique-properties of the 3-space geometry, it is the concept of the matrix-grid of points to represent his 3-space graphics on a flat-sheet of paper, thus, making it possible for him for the first-time to represent hidden-views in all his 3-space graphics; the 2-space matrix-grid of points being such that sufficient-space-points exist for drawing lines that overlap at the minimum within a given 'patch' of the 2-space drawing on a flat-sheet

of paper, i.e. in the black-square OCGA the overlapping line-segments are FG, FE, and FH, it being understood that the geometric-gestalt of points of the black-square OCGA as-a-whole is left intact when viewed by the 'observer' using his inner-eye. The transformed-flatlander manifest the 3-space geometric-forms in physical- and graphic-form via the use of his inner- and outer-senses, or as we say, he uses his inner- and outer-eye to bring into mental-focus the geometric-gestalts of the geometric-forms. The 'isomorphism' of geometric-forms of 3-space then correspond to the projected thought-forms of inner-space of the transformed-flatlander.

Now the remarkable property of hidden-views in the 3-space graphics occurs because of the fundamental-principle that segments with points lying in different face-boundaries of the black-cube as depicted in the graphic-form can be used to represent hidden-views; the same principle holds true for the boundaries of the black-squares that belong to the black-cube, but in this case, the 2 points of a segment must not lie in the plane of a face of any 1 of the black-squares—like say, the segment OF representing 1 of the diagonals of the black-cube, then all the interior-points of this principal-diagonal can be represented as dashed-lines denoting that the 'interior' of the segment OF lies within the interior of the black-cube.

One more basic-property is required for our transformed-flatlander to view 3-space geometric-forms, i.e. depth-perception 'projected' within the 2-space graphic-drawings on a flat-sheet of paper depicting the 3-space geometric-forms. Our transformed-flatlander 'knows' from using his new-synesthetic-sense that 3-space geometric-forms have 'depth', this depth-perception 'ability' he has developed with the aide of his inner-eye to mentally-focus, in such-a-way, that he psychologically-sees and forms the depth-gestalts within his inner-self, which he then projects within the 2-space matrix-grid of points those geometric-graphic-forms lying on a flat-sheet of paper. The seeing is NOT in the 2-space graphic-representation, but lies instead within his so called 3-space inner-self. As we would say in spaceland: 'he imagines the depth in the 2-space graphic-representation with his inner-eye'. This inner-space-perception of outer-space solids is created by the inner-self to adjust for the camouflage or distortion inherent in the limits of the outer-sense-organs of sight.

But as the transformed-flatlander saw in his former 2-space 'experience' the boundary-points A and B of the hidden-edges OA and OB of the black-square OAEB. Then in 3-space he sees for the first-time in each of the 3 ½-visible black-squares about the point O, that is, the visible-edges of the 3 black-squares that meet at the point O as depicted in the 3-space graphic of the black-cube on plate-I, the following situation: in the hidden-view of the face of the black-square OBHC he sees the 2 visible-edges CH and HB bounding a portion of the face of this black-square; in the hidden-view of the face of the black-square OCGA he sees the 2 visible-edges CG and GA bounding a portion of the face of this black-square; in the hidden-view of the black-square OAEB he sees the visible-edges BE and EA bounding a portion of the face of this black-square. If we delete the 3 black-squares about the point F as well as their faces together with all the interior-points of the black-cube, then our transformed-flatlander will see the 3 former hidden-views of the faces of the black-squares about the point O as well as the edges of these black-squares about the point O, i.e. all the faces and boundaries of the black-squares that meet at the point O will now be visible. The positive-portions of the coordinate-axes lying on the coordinate-planes of the 3 black-squares about the point O are now visible.

Just as the flatlander's 2-space experience of the 2 invisible-portions of the positive-coordinate-axes, i.e. the y- and x-axes, so in 3-space too, the 3 positive-portions of the x-, y-, and z-axes about the point O are hidden-views of the black-cube.

As our transformed-flatlander explores this new dimension he will discover many remarkable new-relationships that have no-existence in his flatland-geometry; many similarities will be found as well as the differences.

Without going into repetition again, we conclude this section by stating that our transformed-flatlander will discover the principle of the single-oblique-projection in the 3-space perception, noting the differences and similarities to his 2-space planimetry. He will find to his amazement that the scale-distortion-factor becomes O when viewing 1 of the 2-space squares, like the plane on which the black-square OCGA lies, also including of course the black-square BHFE, and with the oblique-views of the planes of the other remaining black-squares distorted in the 3-space graphics and and having the appearance of parallelograms.

Iab. THE SPACELANDER'S PROJECTION INTO THE 4th DIMENSION

7. OBLIQUE-SYMMETRY. HIDDEN-VIEWS IN THE HYPERSPACE-GRAPHICS. As we had projected our flatlander into the spacelander's 3-space. Let us now take the spacelander and project him into the hyperspacelander's 4-space giving him the powers of double-depth-perception , what we in spaceland would call 'hyperstereometry' of the 4-space geometry. Further, we assume that the transformed-spacelander has been endowed with a 4-space 'energy-form' enabling him to actualize in 4-space the double-objects or supra-symbol-objects, which we in spaceland call hypersolids. The transformed-spacelander using his new founded multidimensional inner-eye can manipulate 4-space mental-images into geometric-forms and project his 4-space ideas into graphic-forms using the 3-space flat-hypersurface to represent the ideal-hypersolids of the 4-space geometry. In the 4-space 'graphics', the transformed-spacelander will use the single-oblique-projection graphic-representation. We assume that the transformed-spacelander's synesthesic-reasoning abilities have been extended in order to study and develop the geometry of 4-space ideal-hypersolids. Using his inner-eye on a 2nd-level of focusing, then what was to him formerly a single 3-space viewpoint of a cube, seen as-a-whole, now becomes in 4-space 1 of the 3-space boundaries of a hypercube.

As a last point of consideration before we discuss the transformed-spacelander's new 4-space 'experiences', we can consider that the transformed-spaselander has actualized the single-oblique-projection of the hyperspace-graphics, whereas, we in 3-space using the double-oblique-projection of the hyperspace-graphics have quasi-actualized the 4-space 'perceptics' with a high degree of certitude. The student should bear this in mind when he considers the SOP and DOP aspects of the 4-space graphics.

Using the figure of the canonical-hypercube as shown on plate-I, suppose our transformed-spacelander were to lie in the hyperspace of the canonical-hypercube and 'obliquely' at a little distance from above the point F', that is, the oblique-symmetric -viewpoint of the canonical-hypercube as shown on plate-I. A subtle-change occurs in the transformed-spacelander's new found synesthesic-sense. He sees the point F' as he did with a similar point F in the 3-space black-cube, for now he sees that whereas before was the line-segment EF, that the 2-space point E had traced-out in moving through a distance of 1-unit perpendicular to the flatlander's plane OAE, has now become the 'red-square' FF'E'E in the motion of the line-segment at FE to its position at F'E', that is, has moved perpendicular to the hyperplane of the black-cube through a distance of 1-unit in hyperspace. The red-square FF'E'E has 1 black-edge of its boundary lying in the hyperplane of the black-cube—this red-square has all of its interior-points lying outside of the hyperplane of the black-cube, and intersects the black-cube in the line-segment FE. The point F has moved to its position at F' in hyperspace. The red-edge FF' of the red-square FF'E'E is perpendicular to the hyperplane of the black-cube at the point F as well as being perpendicular to the black-cube at the point F, the point F is the only point of the line-segment FF' lying in the hyperplane of the black-cube.

The 3 faces of the black-squares of the black-cube which meet at the point F, now become part of the boundary of the red-cubes in the hyperplanes F'FEA, F'FEB, and F'FGA, that is, for the 3 red-cubes that meet at the point F', each will contain a face of a black-square. The transformed-spacelander now discovers a remarkable relationship: a 4th hyperplane parallel to the hyperplane of the black-cube and passing through the point F' intersects the 3 hyperplanes of the red-cubes at the point F' in the 3 red-faces of the red-squares belonging to the red-cubes that meet at the point F'. At the point O' containing the intersection of the parallel-hyperplanes with the 3 hyperplanes of the red-cubes that meet at this point, will intersect the hyperplane of these 3 red-cubes in the faces of the red-squares that meet at the point O'. In the 4th hyperplane parallel to the hyperplane of the black-cube, we will then have the 4th red-cube that meets the other 3 red-cubes at the point F', i.e. 4 red-cubes that meet at the point F' only in this single-point—the 4th red-cube being O'C'G'A'-B'E'F'E'.

The transformed-spacelander sees for the first-time the 4 visible-cubes about the point F'. The transformed-spacelander sees for the first-time the 3-space interiors of each of the 4 red-cubes about the point F', i.e. what was the 'interior' of his 3-space black-cube now becomes the boundary of a higher-figure in 4-space. He sees the 3-space boundaries of the 4 red-cubes about the point F', that is, all vertices, edges, faces, and interiors of these 4 red-cubes about the point F'.

He finds that the new barrier that is impenetrable are the 3-space boundaries of the hypercube. Our transformed-spacelander discovers the 3-space 'parallel' between his spaceland-experience of 'seeing' and the new 4-space perception. He sees the 4 visible red-cubes about the point F' and the 4 invisible-cubes about the point O in the SOP-graphic of the canonical-hypercube. We have at the point O 4 invisible cubes that meet in this 1 point, i.e. 3 invisible red-cubes and 1 invisible black-cube. By letting our transformed-spacelander go around the real-hypercube to view the different viewpoints of the cells of this hypercube, shows how valid his 'logic' was in the SOP-graphic-representation of the ideal-hypercube. He sees that the 4 invisible-cubes at the corner of the hypercube at the point O, now become the new 4 visible-cubes of the hypercube from his new position in hyperspace in viewing the real-hypercube. The transformed-spacelander discovers that the principle of oblique-symmetry holds valid in the SOP-graphics.

The transformed-spacelander discovers another remarkable relationship that holds valid in his new 4-space experience. His old 'definition': that the interior-points of certain segments can be used to represent hidden-points in the interior of the hypercube, thus making it possible to represent hidden-views for the interior-points of certain segments, that is, in the 4-space SOP-graphics. Our transformed-spacelander reasons that a similar process occurs in 4-space: take any 2 distinct points lying in the 3-space cells of the hypercube, such that no 2 of these points will lie in the same cell, i.e. each of the 2 points of a segment lying 1 in each of any 2 of the 8 cells of the hypercube, then the resulting segment will have all of its interior-points lie within the hypercube; likewise, any 2 distinct points of a segment whether at the vertices, edges, or faces of the hypercube, will have all of its interior-points lie within the hypercube, providing that these 2 distinct points do not all lie in 1 hyperplane.

Just as 'portions' of the boundaries of the invisible black-squares of the black-cube at the point O can be visibly seen, that is, in the hyperplane of the black-cube, then ½ of the boundaries in each of the 3 invisible black-squares about the point O are visible, as we had already seen from a study of this in a previous section. So in hyperspace we should have a similar relationship.

Our transformed-spacelander will discover that of the 4 cubes at the point O that are invisible in the graphic-drawing, ½ of the boundaries of each of the 4 cubes will be visible, i.e. in the black-cube the 3 visible-boundaries are the faces of the black-squares AEFG, BHFE, and CGFH together with the black-squares themselves; in the red-cube OBEA-O'B'E'A' the 3 visible-boundaries are the faces of the red-squares AEE'A', BB'E'E, and O'A'E'B' together with the red-squares themselves—this includes of course the black-edges that belong to these red-squares; for the remaining other 2 red-cubes about the point O the results are similar.

The transformed-spacelander will discover that if he deletes the 4 visible red-cubes about the point F' as well as the interior-points of the hypercube, then he will see all of the 4 cubes lying in the coordinate-hyperplanes about the point O, i.e. the 4 invisible-cubes about the point O will now become visible, that is, not only their 2-space boundaries but also all the interior-points in each of the 4 cubes about the point O; for example, in the black-cube we would see from our 4-space viewpoint all of its interior-points as well as all of the 6 faces of this black-cube—we use 2nd-level inner-eye focusing here, something like multiple lens-focusing in 'optics'. We could call this inner-eye focusing-process 2F for 4-space perceptics and 1F for 3-space perceptics. We could also, somewhat crudely, call the inner-eye multiple-focusing-process, gestalt-resolvents within the SOP- and DOP-graphics.

Since our transformed-spacelander can now 'visualize' the hypercube with ease and with his inner-senses 'feel' as well its 3-space boundaries, he decides to cut-apart the hypercube in various ways, then putting it together again to obtain new-knowledge of this strange new 4th-dimension. Later on, his ultra-technological-discoveries will astonish him, for he now is a growing-child hyperspacelander. He might even infer that all matter-states lie within given energy-bands of electromagnetic-energy-fields, that is, something like fields within 'fields' within "fields" within ... A matter-spectrum of EM-units would correspond to the different energy-bands for the materializations of the different density-levels of 'matter'. Our transformed-spacelander could not infer the above statements until he had 'actually' created the possible thought-forms with the aide of his inner-eye and inner-senses, then and only then, could he create the supra-

symbol-objects of hyperspace, thus making the 3-space symbol-objects a mere 'shadow-world' of shadow-objects no longer considered as being 'solid' from the higher-space viewpoint.

The student of this treatise could go on giving many illustrations of other properties of the double-oblique-projection of the canonical-hypercube which are not covered in this treatise. The general-method of the DOP-graphics has applications in other branches of mathematics. Those interested in the theory of 2 complex-variables and <u>quadratic-hypersurfaces</u> will find much to be developed with the aide of the DOP-graphics of the canonical-hypercube. Those in the field of combinatorial-analysis could for the first-time represent the hypersolid-forms of the 4-space <u>partitions</u>, that is, the 4-space <u>graphs</u> of the partitions of the natural-numbers. Applications are almost infinite.

In the chapters that follow we will develop the 4-space DOP-graphics. In this treatis we will make a few applications of the visual-hypersolid-geometry to the 4-space hypersolid-analytic-geometry, 3-space Non-Euclidean-geometry, and an illustration of the methods of the calculus as applied to the DOP-graphics.

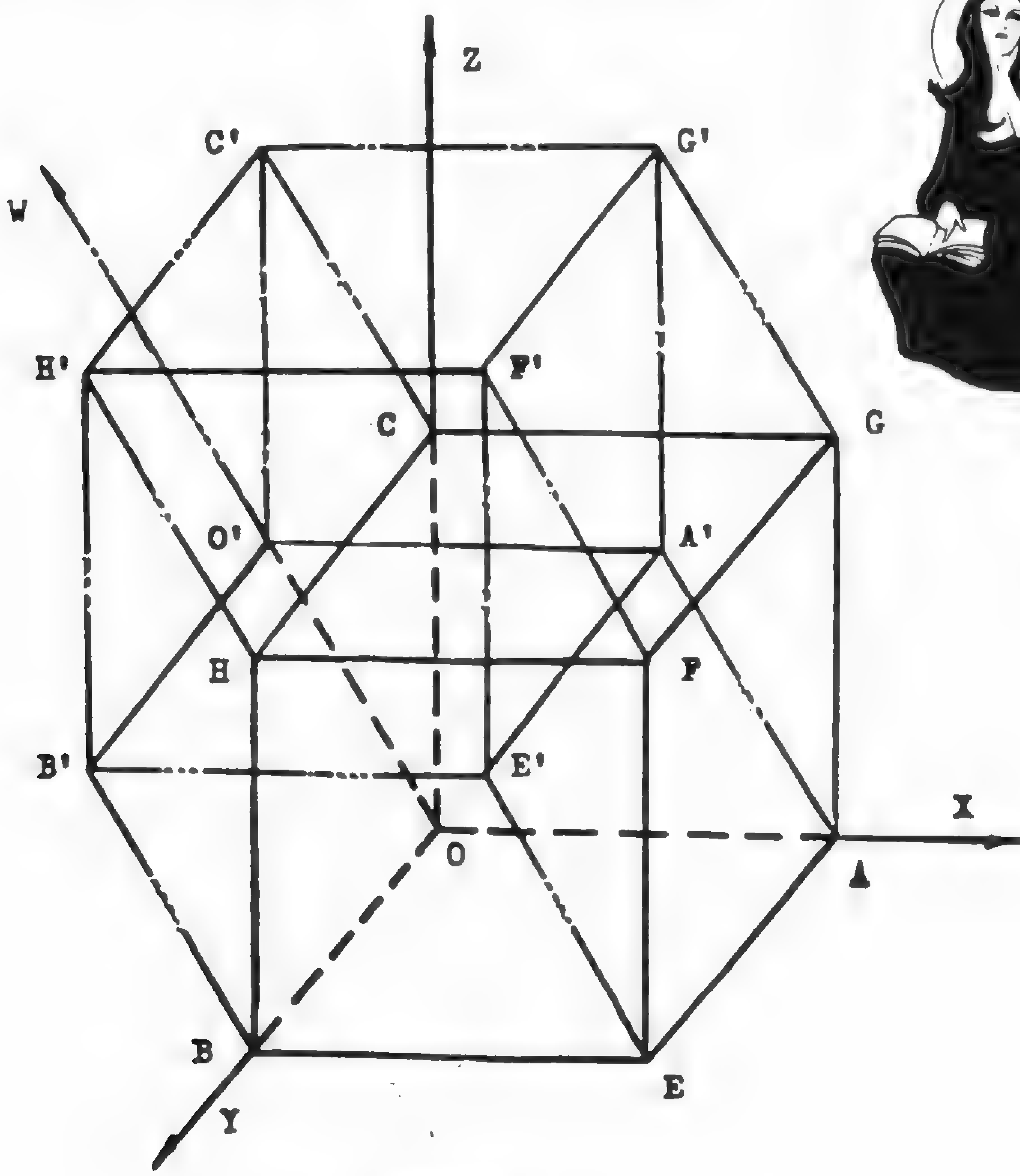


Plate-I

CANONICAL-HYPERCUBE

Scale and Angle-Codes:

OA = 1.75"
OC = 1.75"
OB = 1.25"
OO'= 1.50"

angle COO' = 30°
angle COB = 140°

CHAPTER II

I. HYPERPLANES IN HYPERSPACE

1. DETERMINATION OF A HYPERPLANE—Definitions. A hyperplane consists of the points that we get if we take 4 points, not points of 1 plane, all points collinear with any 2 of them, and all points collinear with any 2 obtained by this process. Given 4 non-coplanar points, A, B, C, and D, the HYPERPLANE ABCD is the hyperplane obtained when we take these points and carry out the process described in the definition.

Theorem 1. If 2 points of a line lie in a given hyperplane, the line lies entirely in the hyperplane; and if 3 non-collinear points of a plane lie in a given hyperplane, the plane lies entirely in the hyperplane.

For the line or plane can be obtained from these points by the process used in obtaining the hyperplane.

It follows that a plane having 2 points in a given hyperplane, but not lying entirely in it, will intersect the hyperplane in the line which contains these 2 points.

Theorem 2. From the points of the figure given in each of the following cases we can obtain just the points of a hyperplane if we take all points collinear with any 2 of them and all points collinear with any 2 obtained by this process:

(1) A plane and a point not in it, or a plane and a line that intersects it but does not lie in it;
(2) 2 lines not in 1 plane;
(3) 3 lines through 1 point but not in 1 plane;
(4) 2 planes intersecting in a line.

We can speak of a line or a plane as one of the things with which we start in the process of obtaining a hyperplane.

It follows from (1) that a line or a plane which do not lie in a hyperplane do not intersect at all, and from (4) that 2 planes which do not lie in a hyperplane cannot have more than 1 point in common.

Given any 4 non-coplanar points of a hyperplane we then have the following theorem.

Theorem 3. If A', B', C', and D' are 4 non-coplanar points of the hyperplane ABCD, then the hyperplane A'B'C'D' is the same as the hyperplane ABCD.

In hyperspace of the hypersolid-geometry, a given configuration may contain portions of many distinct hyperplanes. The 4 conditions listed above in Th. 2 can be used to specify or determine the distinct hyperplanes of such a hyperspace-configuration.

To stress again the importance of theorem 2 we can now say that the hyperplane obtained in each case is the only hyperplane that contains the given figures.

3 non-collinear points can be points of 2 different hyperplanes. The intersection of the hyperplanes will then be the plane of the 3 points (see Art. 4 Th. 2).

Actually we get all the points of ordinary-space by taking 4 non-coplanar points, all points collinear with any 2 of them, and all points collinear with any 2 obtained by this process. The space of our experience will therefore be regarded as a hyperplane.

Although hyperplanes are unlimited in extent, we will represent them as parallelopipeds.

II. SPACE OF 4 DIMENSIONS

2. RESTRICTION TO 4 DIMENSIONS. A SPACE OF 4 DIMENSIONS consists of the points that we get, if we take 5 points not points of 1 hyperplane, all points collinear with any 2 of them, and all points collinear with any 2 obtained by this process.

The preceding theorem can be put into a more precise-form as follows:

Theorem 4. We get all points if we take any 5 points not points of 1 hyperplane, all points collinear with any 2 of them, and all points collinear with any 2 obtained by this process.

In this treatise all geometrical-relationships are assumed to lie in 4-space.

3. PENTAHEDROIDS—Interior. THE COLLINEAR-RELATION. INTERSECTION WITH A PLANE. A PENTAHEDROID consists of 5 points not points of 1 hyperplane, and the edges, faces, and interiors of the 5 tetrahedrons whose vertices are these points taken 4 at-a-time.

The 5 points are the VERTICES, the edges and faces of the tetrahedrons are EDGES and

FACES of the pentahedroid, and the interiors of the tetrahedrons are its CELLS. Any 5 points, not points of 1 hyperplane, are the vertices of a pentahedroid.

At times we shall speak of the vertices, edges, and faces of a cell, but it should be understood that a cell of the pentahedroid is the interior of a tetrahedron and does not include the tetrahedron itself.

The INTERIOR of a PENTAHEDROID consists of the interiors of all segments whose points are points of the pentahedroid, except of those segments whose interiors also lie in the pentahedroid.

A point is said to be COLLINEAR with a pentahedroid when it is collinear with any 2 points of the pentahedroid. The collinear-relation holds true for all the interior-points of the pentahedroid.

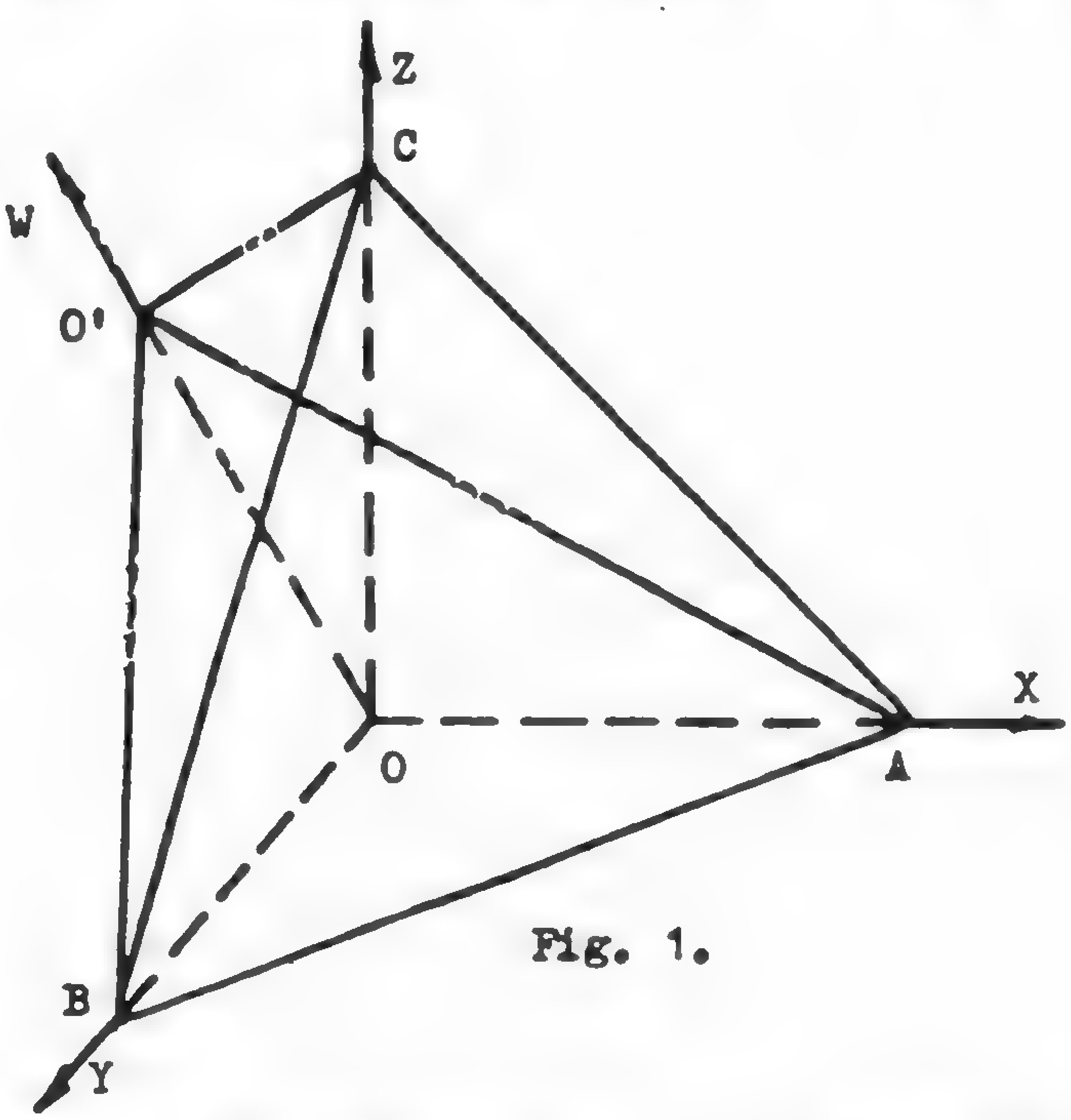


Fig. 1.

3a. GRAPHIC-CONSTRUCTION of the PENTAHEDROID—Fig. 1. In the figure of the canonical-hypercube given on plate-I, take in the hyperplane of the black-cube that portion made-up of the 4 non-coplanar points O, A, B, and C, then form the coordinate-tetrahedron OABC; in the hyperplane of the red-cube OAEB-O'A'E'B' take that portion made-up of the 4 non-coplanar points O, A, B, and O', then form a 2nd coordinate-tetrahedron OABO', these 2 coordinate-tetrahedrons will intersect in the common-base OAB; now draw a line-segment from the point O' to the point C.

In the unit-coordinate-pentahedroid as constructed, we have 4 red-tetrahedral-cells and 1 black-tetrahedral-cell; the 4 red-cells being OACO', OBCO', OABO', and O'ABC; the single black-cell being OABC.

Assume, for a moment, that the positive-portion of the w-axis vanishes. Then we shall have only the 3-space coordinate-tetrahedron OABC, which in the graphic-form, has 3 hidden-faces and 1 visible-face ABC together with all of the hidden-points in the interior of this tetrahedron. In OABC, the visible-edges are AB, BC, and CA belonging to the 3 hidden-faces OAB, OBC, OCA, respectively; these 3 visible edge-boundaries also form a triangle ABC belonging to the visible-face ABC.

A similar result occurs also for the coordinate-pentahedroid as-a-whole: 4 hidden-views of the cells OACO', OBCO', OABO', and OABC, with 1 visible-view being in the hyperplane of the tetrahedral-cell O'ABC. Likewise, in the pentahedroid, we have 4 visible-faces ABC, ABO', BCO', and CAO', with 1 of each of these belonging to 1 of each of the 4 hidden-cells of the pentahedroid; in the hidden-views for each of the tetrahedrons, excepting the black-tetrahedron OABC, we have in each, 3 hidden-faces and 1 hidden-cell—the only visible-cell being that of the tetrahedron O'ABC together with its 4 visible-faces.

In the graphic-forms, when we do not use a rectangular-coordinate-system, we shall speak of the coordinate-tetrahedron OABC, in 1 of 3-ways, i.e. 'the tetrahedron OABC',

or 'a tetrahedron OABC', or 'any tetrahedron OABC', depending on the sense it is used in a given context; similarly, the coordinate-pentahedroid O'OABC, will be spoken of in 1 of 3-ways, i.e. 'the pentahedroid O'OABC, or 'a pentahedroid O'OABC', or 'a pentahedroid O'OABC', depending upon the sense it is used in a given context.

Hereafter, all 4-space geometric-figures as constructed, will be made-up from the points of the coordinate-hyperplanes and the hyperplanes that are parallel to these coordinate-hyperplanes. The extended geometric-figures will be discussed in another chapter.

Now, in the 4-space viewpoint, you must use your inner-eye, in such-a-way, that you mentally see as-a-whole all of the interior-points of the tetrahedral-cell O'ABC, and further, with your inner-eye imagine that all the interior-points of tetrahedral-cells with hidden-views are invisible-points. It will take a little while to get a real-mental-feel of the inner-sensing-processes used here to 'visualize' as-a-whole the 4-space geometric-gestalt-forms. Then as you develop the ability to express the geometric-relationships pictorially, the theorems and proofs of the 4-space geometry will become intuitively-understood with extreme-clarity.

Theorem 1. The plane of 3 non-collinear points of a pentahedroid, if it does not itself lie in the hyperplane of 1 of the cells, intersects the pentahedroid in a convex-polygon. (Fig. 2.)

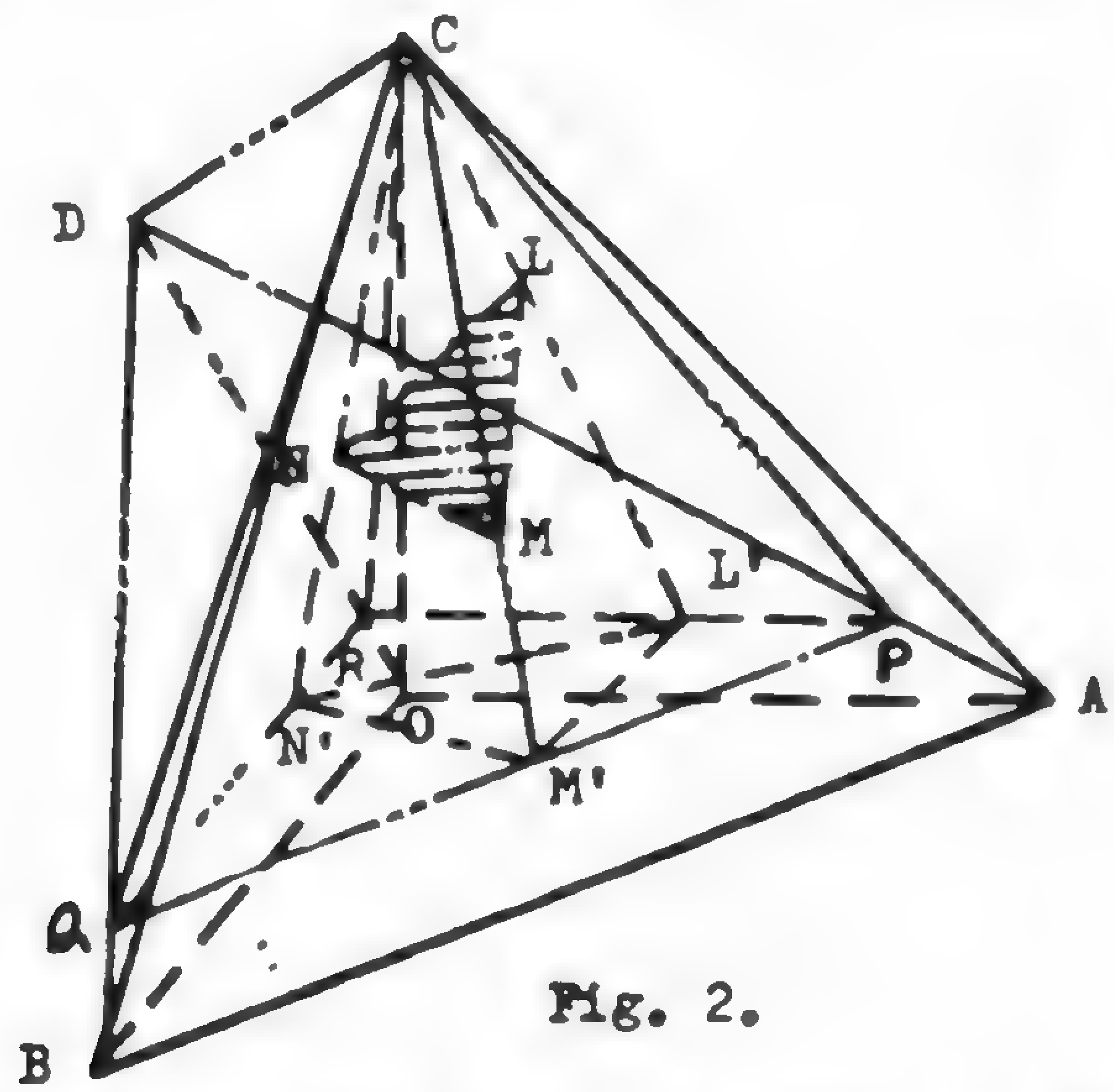


Fig. 2.

We shall prove the theorem for the case when the plane of 3 non-collinear points of the pentahedroid, intersects the pentahedroid in a triangle. For the case when the plane of 3 non-collinear points of the pentahedroid, intersects the pentahedroid in a quadrilateral, then use a proof similar to that given for the triangle.

Given: The pentahedroid OABCD, and any 3 non-collinear points L, M, and N of the pentahedroid not in the hyperplane of 1 cell.

To Prove: The plane of LMN intersects the pentahedroid in a triangle.

Proof: At any point R in the interior of the segment OD near the point O, construct a plane passing through R and intersecting the red-tetrahedron OABD in the triangle PQR, then take the vertex-point C with this triangle and form the pyramid C-PQR; CP, CQ, and CR are the lateral-edges of this pyramid.

Take any point L in the face CPR, this face lies in the cell OACD, and the point L lies in the interior of this tetrahedron; take any point M in the face CPQ, this face lies in the cell DABC, and the point M lies in the interior of this tetrahedron; take any point N in the face CQR, this face lies in the cell OBCD, and the point N lies in the interior of this tetrahedron.

Let the projections of the points L, M, and N from the point C meet in the edges of the red-pyramid C-PQR in the points L', M', N', respectively, i.e. the point L' in RP, M' in PQ, and N' in QR.

Now C is a vertex not in the plane of the given points L, M, and N. L being a point of the tetrahedral-cell OACD, and being in the face of the triangle CPR which lies in the

interior of this tetrahedron, then the $\frac{1}{2}$-line CL meets the hyperplane of the opposite-cell OABD in the face OAD at the point L' which is the projection of L from $\emptyset$. L' is a point of the tetrahedron OABD (Th. 1, and Th. 2 of RSG-II)*. The projections of the other 2 $\frac{1}{2}$-lines CM and CN follows in the same manner as that given for L, it being understood from the construction given in Fig. 2. Now the projections of the 3 given points L, M, and N from C are the points L', M', and N'. These last 3 points are not collinear; for, if they were, the plane determined by their common-line and C would be the plane containing the 3 given points and the vertex C. They are not all in the plane of any one face of the tetrahedron OABD; for, if they were, L, M, and N would lie entirely in the hyperplane determined by this face and C, and the given plane would lie entirely in this hyperplane. Therefore the plane L'M'N' intersects the tetrahedron OABD in a triangle PQR (Th. 3 of RSG-II). This triangle PQR is the base of a pyramid CPQR with the vertex at C which lies entirely in the hyperplane CPQR, and is the intersection of the hyperplane CPQR and the pentahedroid (see Art. 7, Th. 1). The points L, M, and N are points of the pyramid C-PQR, and the plane LMN lies entirely in the hyperplane of this pyramid. The intersection of the plane and the pentahedroid is the same as the intersection of the plane and this pyramid; it is a triangle (Th. 4 of RSG-II). Therefore the plane LMN intersects the pentahedroid in a triangle. (Q.E.D.)

The reader should note, for example, that the plane of LMN passing through the point M of the red-pyramid C-PQR intersects the face CPQ of this pyramid in a line-segment, and since the triangle CPQ lies also in the tetrahedral-cell OABC, the plane of LMN intersects this cell in the same line-segment.

Since the pentahedroid has only 5 cells, the intersection can only be a triangle, a quadrilateral, or a pentagon.

Theorem 2. Any line intersecting a cell of a pentahedroid will intersect the pentahedroid at least in a 2nd point, and any $\frac{1}{2}$-line drawn from a point O of the interior of the pentahedroid will intersect the pentahedroid.

Theorem 3. Any plane intersecting a cell of a pentahedroid, if it does not itself lie in the hyperplane of this cell, or any plane containing a point O of the interior of the pentahedroid, will intersect the pentahedroid in a convex-polygon.

*Note: The star-asterisk above refers to the review-section of solid-geometry given at the beginning of this treatise. Whenever a theorem from the solid-geometry is used as part of the proof for a theorem of the hypersolid-geometry, we will simply refer to the review-section listing the theorems needed from the solid-geometry. The review-listings of the theorems needed will be grouped by chapter-headings to facilitate easy reference.

4. INTERSECTION OF A PLANE AND A HYPERPLANE AND OF 2 HYPERPLANES.

Theorem 1. If a plane and a hyperplane have a point O in common, they have in common a line through O.

Proof: We construct a pentahedroid with a cell lying in the given hyperplane and containing the point O. The given plane intersects the pentahedroid in a convex-polygon and the given hyperplane in the line which contains one side of the polygon.

Theorem 2. If 2 hyperplanes have a point O in common, they have in common a plane through O.

Given: The 2 hyperplanes OABC and OBCO' with the point O in common.

To Prove: The plane OBC through O is common to the 2 given hyperplanes.

Proof: Use the figure of the canonical-hypercube on plate-I for the graphic-representation of this theorem, that is, a part of this figure.

Any plane OAC through O in the hyperplane OABC intersects the hyperplane OBCO' in the line OC through O by Th. 1; a 2nd plane OAB in the hyperplane OABC intersects the hyperplane OBCO' in the line OB through O by Th. 1; the 2 lines OC and OB through O are common to the 2 given hyperplanes, and these 2 lines determine the plane OBC. Therefore the hyperplanes intersect in a plane (see Art. 1). (Q.E.D.)

3 hyperplanes having a point in common have in common at least 1 line, a line lying in 1 hyperplane and in the plane of intersection of the other 2. 3 hyperplanes may also have a plane in common.

Theorem 3. 2 planes which do not lie in 1 hyperplane contain a set of lines, 1 and only 1 through each point of either plane which is not a point of the other plane, and any 2 of these lines coplanar.

Given: The planes FEA and F'H'C' which do not lie in 1 hyperplane.

To Prove: Any 2 lines FG and F'G' are coplanar and that each line passes through a point of 1 and only 1 of the given planes.

Proof: Use part of the figure of the canonical-hypercube on plate-I for the graphic-representation of this theorem.
Let G be a point of FEA which is not a point of F'H'C', and let G' be any point of F'H'C' which is not a point of FEA. The hyperplane F'H'C'G' determined by F'H'C' and G intersects FEA in the line FG, and the hyperplane FEAG' determined by FEA and G' intersects F'H'C' in the line F'G' (Th. 1). The lines FG and F'G', each lying in both hyperplanes lie in the plane F'FG which is the plane of intersection of the hyperplanes FEAG' and F'H'C'G. Further, no 2 lines lying in 1 of the given planes and coplanar with lines in the other can intersect in a point which is not common to the 2 given planes: for, if they did, both of them and the entire plane in which they lie would lie in the hyperplane determined by their point of intersection and the other given plane. Therefore the lines FG and F'G' are coplanar and each line passes through a point of 1 and only 1 of the 2 given planes. (Q.E.D.)

The planes FEA and F'H'C' are covered with these lines, and can be considered to be made-up of these lines. We shall call them the LINEAR-ELEMENTS of the 2 planes. When the 2 planes have a point in common, the linear-elements all pass through this point. If any plane intersects the 2 planes in lines, these lines are linear-elements.

5. OPPOSITE-SIDES OF A HYPERPLANE. ½-HYPERSPACES. In the figure of the canonical-hypercube on plate-I, suppose we take the hyperplane of the black-cube. We can then say that a hyperplane divides the rest of hyperspace, just as a plane in a hyperplane divides the rest of the hyperplane (see Th. 5 of BSG-II). We can speak of the OPPOSITE-SIDES OF A HYPERPLANE, and of a ½-HYPERSPACE. We may have, for example, the ½-HYPERSPACE OABC-O' lying on one-side of the hyperplane OABC, that is, on that side of the ½-line at O of OO' produced. The other-side of the hyperplane OABC will be the ½-hyperspace lying on that side of the negative w-axis. The hyperplane OABC does not belong to either of the ½-hyperspaces, it separates the opposite ½-hyperspaces of this hyperplane.
If 2 hyperplanes intersect, the opposite ½-hyperplanes of each which have the plane of intersection for a common-face lie on opposite-sides of the other.
Given a pentahedroid, each of the 5 tetrahedrons determines the cell of a ½-hyperspace which contains the opposite vertex and all points of the interior; and, conversely, if a point lies in all of these ½-hyperspaces it will lie in the interior of the pentahedroid. Every point of hyperspace is a point of at least 1 of these ½-hyperspaces.
We shall at times speak of a pentahedroid as a HYPERSURFACE (see Art.38), and of its interior as a HYPERSOLID. A pentahedroid divides the rest of hyperspace into 2 portions, interior and exterior to the pentahedroid.
In hyperspace a hyperplane divides the hyperspace into 2 parts; whereas, a plane divides a hyperplane into 2 parts, but not hyperspace; whereas, a line divides a plane into 2 parts, but not a hyperplane.

III. HYPERPYRAMIDS AND HYPERCONES

6. HYPERPYRAMID. INTERIOR OF A HYPERPYRAMID. Figures in hyperspace which correspond to the polyhedrons of geometry of 3 dimensions are called POLYHEDROIDS. We shall not define this term, except to say that a polyhedroid consists of VERTICES, EDGES, FACES, and CELLS. The cells being the interiors of certain hyperplane-polyhedrons joined to one another by their faces so as to enclose a portion of hyperspace, the INTERIOR OF THE POLYHEDROID. We shall apply the term 'polyhedroid' only to certain simple-figures which we shall define individually. The pentahedroid is the simplest polyhedroid.

A HYPERPYRAMID consists of a hyperplane-polyhedron enclosing a portion of its hyperplane, and a point not a point of this hyperplane, together with the interior of the polyhedron and the interior of the segments formed by taking the given point with

the points of the polyhedron. The point is the VERTEX, and the interior of the polyhedron is the BASE. The meaning of other terms used in connection with the hyperpyramid may be readily inferred from the definitions of Art. 3 and from the definitions of pyramids used in the solid-geometry.

The INTERIOR OF A HYPERPYRAMID can be defined as consisting of the interiors of the segments formed by taking the vertex with the points of the base, but in the case of a convex-hyperpyramid the interior of any segment whose points are points of the hyperpyramid will lie entirely in the interior of the hyperpyramid unless it lies entirely in the hyperpyramid itself. No line can intersect a convex-hyperpyramid in more than 2 points unless it lies in the hyperplane of 1 of the cells, and any $\frac{1}{2}$-line drawn from a point O of the interior will intersect the hyperpyramid in 1 and only 1 point.

In Fig. 1, we have a hyperplane-tetrahedron OABC enclosing a portion of its hyperplane together with the interior of this tetrahedron. Take the point O', not a point of this hyperplane, with the points of the tetrahedron OABC. The point O' is the vertex, and the interior of the tetrahedron OABC is the base. The interior of this convex-hyperpyramid is defined as consisting of the interiors of the segments formed by taking the vertex O' with the points of the base OABC.

To make crystal-clear the way in which we visualize 'segments with interior-points lying in the interior of the hyperpyramid', and <u>in</u> the hyperpyramid, refer to Fig. 6b of Art. 8. For example, the segment DP has <u>all</u> its interior-points lying in the interior of the hyperpyramid D-OABC; whereas, the segment DR has <u>all</u> its interior-points lying in the hyperplane of the cell DOAB, and therefore these interior-points lie <u>in</u> the hyperpyramid.

Take any point Q in the interior of the triangle CDR, this point lies in the interior of the hyperpyramid D-OABC (see Art. 7); take the $\frac{1}{2}$-line of DQ produced from the point Q, then this $\frac{1}{2}$-line will intersect the hyperpyramid in the point P lying in the hyperplane of the cell OABC, and <u>only</u> in the point P of this cell; the point P lies in the hyperpyramid-base. The segment CR lying in the hyperplane of the base OABC has all its interior-points lying in this base, and therefore <u>all</u> the interior-points of this segment lie <u>in</u> the hyperpyramid.

7. HYPERPLANE-SECTIONS OF A HYPERPYRAMID. A hyperpyramid or any polyhedroid can be cut by a hyperplane in a SECTION which divides the rest of the polyhedroid into 2 parts lying on opposite-sides of the hyperplane (Art. 5).

The sections of a polyhedroid are polyhedrons whose faces are the sections of the cells of the polyhedroid made by the planes in which the hyperplane intersects the hyperplane of the cells. The group of theorems given in this section for the hyperplane-sections of a hyperpyramid or pentahedroid are proved by considering the plane-sections of their cells.

Theorem 1. A section of a convex-hyperpyramid made by a hyperplane containing the vertex, is a convex-pyramid whose base is the corresponding plane-section of the base of the hyperpyramid. In the case of a pentahedroid this applies to any vertex. When 1 vertex of a pentahedroid lies in a hyperplane and 2 vertices of the opposite-cell lie on each side of the hyperplane, the section will be a quadrilateral-pyramid. In all other cases the section of a pentahedroid by a hyperplane containing 1 vertex and not containing a cell will be a tetrahedron.

We shall give 2 graphic-forms for this theorem for the case when the convex-hyperpyramid is a pentahedroid. The student can make-up his own graphic-forms when the base of a convex-hyperpyramid is a convex-pyramid other than the tetrahedral-type—these will be very similar to the graphic-forms given in this text. We shall separate the theorem into 2 parts as follows:

Theorem 1a. When the vertex of a pentahedroid lies in a hyperplane and 2 vertces of the opposite-cell lie on each side of the hyperplane, the section will be a quadrilateral-pyramid. (Fig. 3a)

In the graphic-construction of Th. 1a, the hyperplane-section is the hyperplane of a quadrilateral-pyramid D-KLMN. The vertex D lies in this hyperplane, and the opposite-cell to D is OABC. The 2 vertices O and A lie on opposite-sides of the hyperplane of the quadrilateral-pyramid, that is, the line-segment OA intersects the hyperplane of the

quadrilateral-pyramid in the point N, and the point A lies on one-side of each of the 3 faces of the quadrilateral-pyramid that meet at the point N, whereas, the point O lies on the other-side of each of these 3 faces that meet at the point N. This can be readily seen by 'observing' the 3 faces of the quadrilateral-pyramid that meet at the point N and which lie in different cells of the pentahedroid, the points O and A of the line-segment OA will then lie on opposite-sides of each of these faces; and, the opposite-sides of a face can only be determined when we are given the hyperplane <u>in</u> which it lies.

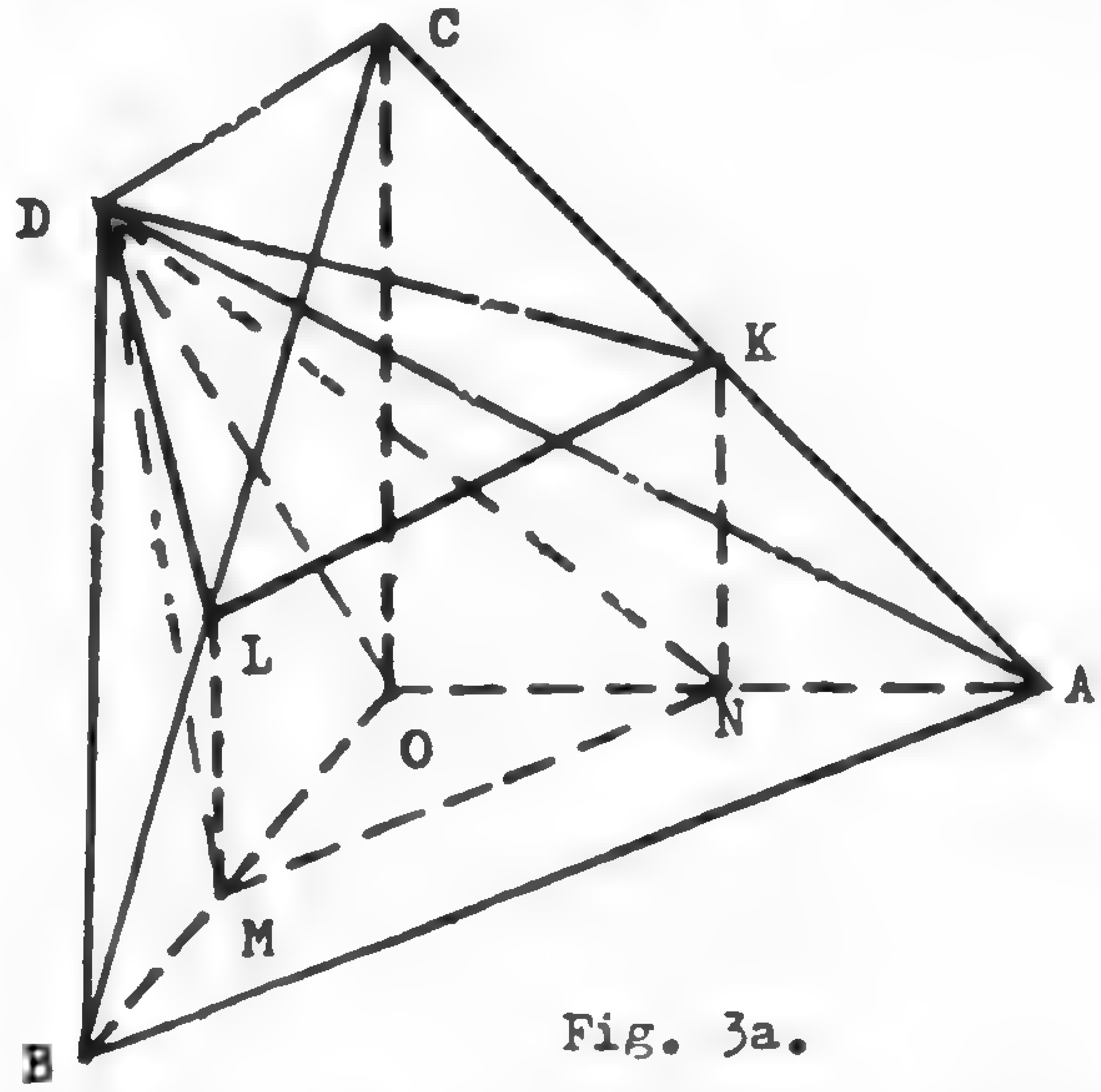


Fig. 3a.

Theorem 1b. In all other cases the section of a pentahedroid by a hyperplane containing 1 vertex an not containing a cell will be a tetrahedron. (Fig. 3b.)

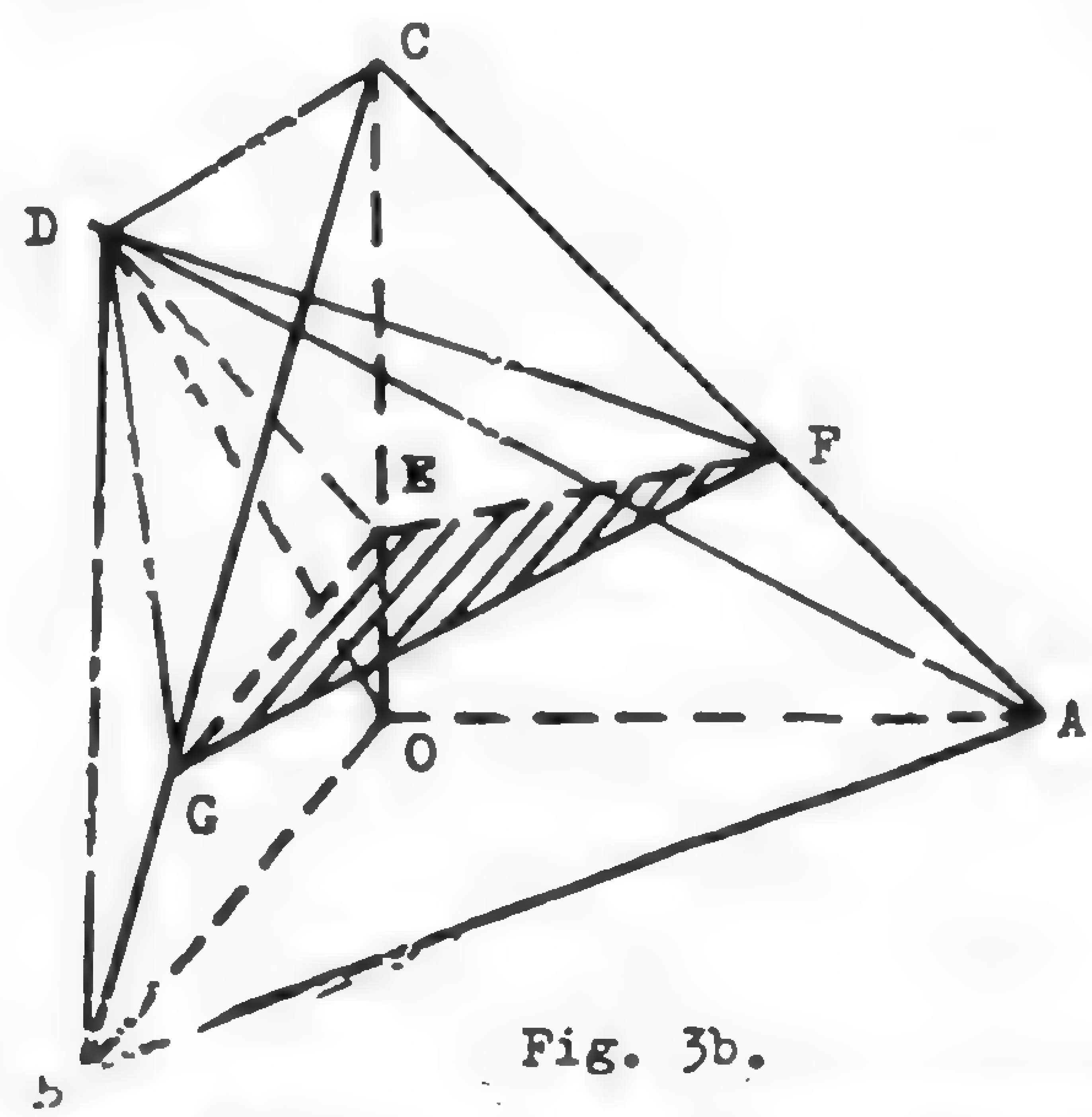


Fig. 3b.

In Fig. 3b, the hyper[illegible]-section is a hyperplane-tetrahedron DEFG.
The student should [illegible] the resemblance to the corresponding theorem in the 3-space sol[illegible]-geometry, th[illegible], in the hyperplane of the black-tetrahedron OABC—the corresp[illegible]nce i[illegible] obvious, that we need not discuss it here. However, the student should compa[illegible] 4-space visualization of the pentahedroid to the corresponding vis[illegible]alizatio[illegible] the 3-space black-tetrahedron, that is, the visible- and hidden-views of the boun[illegible]ies and interiors of the 3- and 4-space graphic-forms of these respective geometric-[illegible]ures.

Theore[illegible]. A hyperplane passing between 1 vertex of a pentahedroid and the opposit[illegible]tetrahedron will intersect the pentahedroid in a tetrahedron. (Fig. 4)

Let a h[illegible]perplane pass between a vertex D and the opposite-tetrahedron OABC. Then the [illegible]yperplane will intersect the pentahedroid OABCD a tetrahedron EFGH.

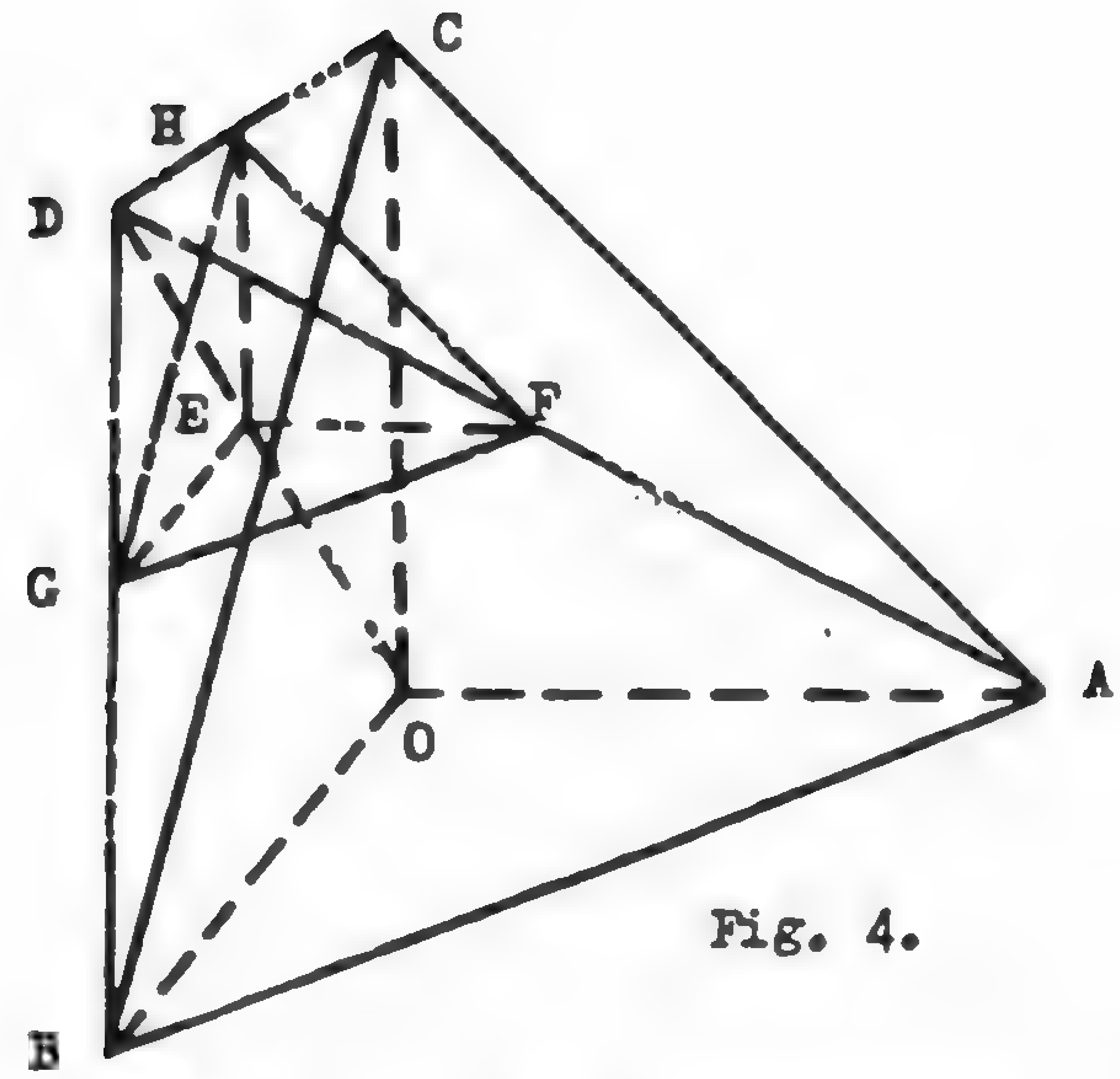


Fig. 4.

Compare the result of this theorem with the corresponding theorem in the 3-space solid-geometry, that is, for the individual tetrahedrons of the pentahedroid. For example, in the tetrahedron OABD, the vertex D lies opposite to the face OAB of this tetrahedron, and a plane passing between the vertex D and the opposite-face OAB will intersect the tetrahedron in a triangle EFG.

A section of a figure in hyperspace is all that we can see in a hyperplane. For, suppose we were in the hyperplane of the red-cell OABD, then we would, see, in our graphic-form, only the hidden-view representation of the base EFG of the red-tetrahedron EFGH; we would see the visible-edge FG, and the 2 vertices G and F, but the edges EF and GE will be hidden-views as well as the vertex E. If we were in the hyperplane of the red-tetrahedron EFGH, then we would see all of the graphic-representation of this tetrahedron, i.e. the visible- and hidden-views.

If we were to take the limit of the sum of all the right-cross-sections of the x-tetrahedrons cut out from hyperplanes passing through all the points of the line-segment OD, i.e. from the O-tetrahedron at D to the opposite-tetrahedron at O, then the limit of the sum of all these x-tetrahedrons would form a pentahedroid—this makes it possible for us to use the calculus with the graphic-forms given here, this will be shown in a later chapter.

Theorem 3. If 2 vertices of a pentahedroid lie on one-side of a hyperplane and 3 on the opposite-side, the section will be a polyhedron in which their are 2 triangles separated by 3 quadrilaterals. (Fig. 5)

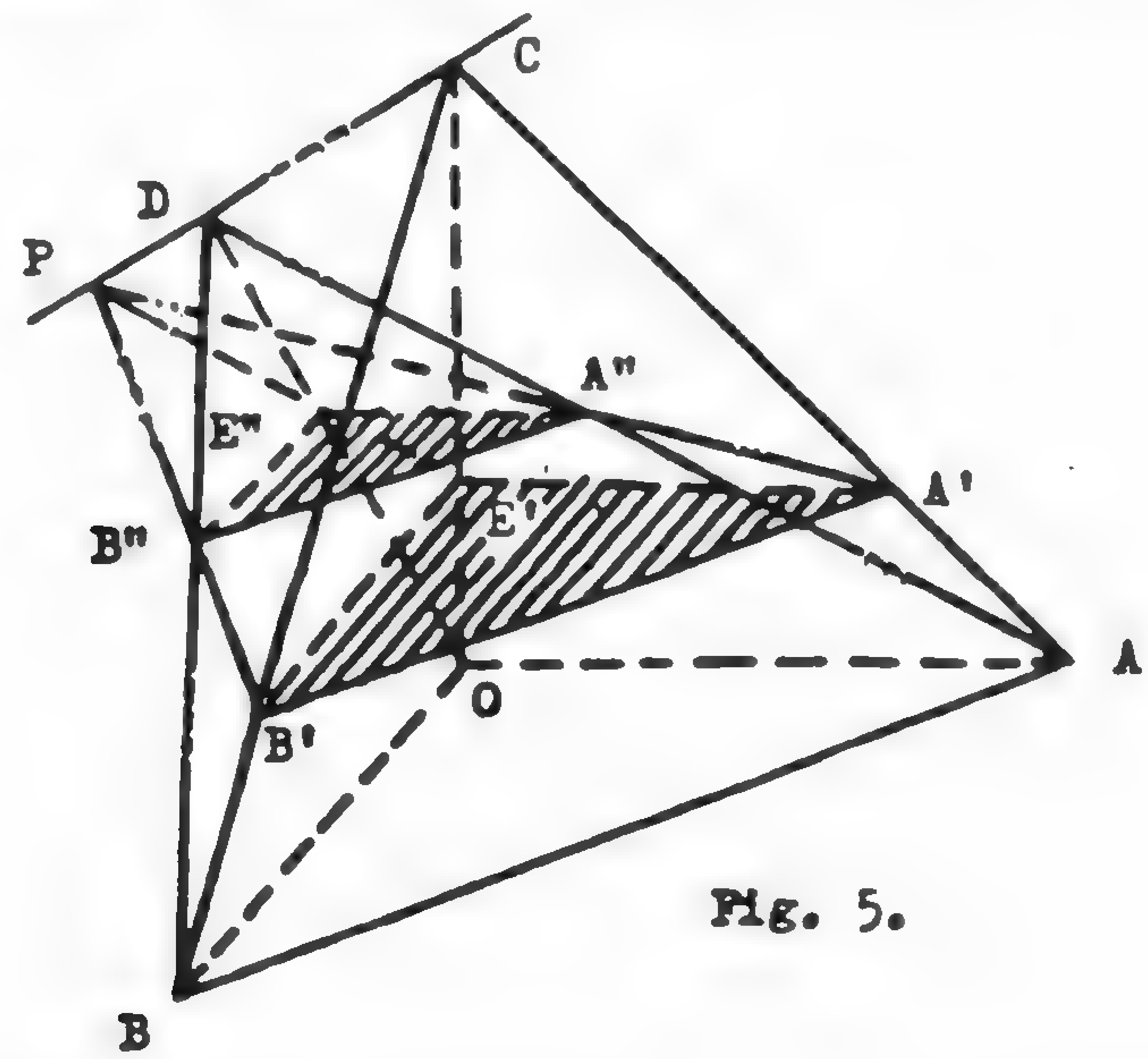


Fig. 5.

Given: A pentahedroid OABCD, C and D on one-side of a hyperplane E'A'B'P, with O, A, and B on the opposite-side.

To Prove: A hyperplane E'A'B'P intersects the pentahedroid OABCD in a polyhedron E'A'B'-E"A"B" in which the 2 triangles E'A'B' and E"A"B" are separated by the 3 quadrilaterals E'E"A"A', E'B'B"E", B'B"A"A'.

Proof: The hyperplanes of the tetrahedrons OABC and OABD are cut by E'A'B'P in planes which pass between the triangle OAB and the points C and D (Art. 5), and which therefore cut these tetrahedrons in triangles. The hyperplanes of the other 3 tetrahedrons are cut by E'A'B'P in planes which pass between their common-edge CD and the opposite-edges OA, AB, and BO, and which therefore cut these tetrahedrons in quadrilaterals. Let the triangles be E'A'B' and E"A"B", the quadrilaterals will be E'E"A"A', E'B'B"E", and B'B"A"A', and the section of the pentahedroid will be E'A'B'-E"A"B".

If E'A'B'P intersects the line CD in a point P, the 3 lines E'E", A'A", and B'B" will pass through P, and the section will be a truncated-tetrahedron. In any case the section will be a figure of this type (see Art.70). (Q.E.D.)

8. DOUBLE-PYRAMIDS. A hyperpyramid whose base is the interior of a pyramid may be reguarded in 2-ways as a hyperpyramid of this kind, the vertex of the base in one-case being the vertex of the hyperpyramid in the other-case.

Thus there are 2 pyramids having themselves a common-base, and we can say that the hyperpyramid is determined by a polygon and 2 points neither of which is in the hyperplane containing the polygon and the other point. Perceived in-this-way the hyperpyramid is called a DOUBLE-PYRAMID.

A double-pyramid consists of the following classes of points:

(1) the points of a convex-polygon, or of any plane-polygon which has an interior, and the points of its interior;

(2) 2 points not in a hyperplane with the polygon, the interior of the segment formed of these 2 points, and the interiors of the segments formed by taking each of these points with the points of the polygon;

(3) the interiors of the triangles formed by taking each point of the polygon with the 2 given points;

(4) the interiors of 2 pyramids each formed by taking the polygon-interior with 1 of the 2 given points.

The interior of the segment of the 2 given points is called the VERTEX-EDGE of the double-pyramid, and the interior of the polygon is the BASE. The interiors of the triangles (3) are called ELEMENTS, and in particular, those elements whose planes contain a vertex of the polygon are LATERAL-FACE-ELEMENTS or LATERAL-FACES of the double-pyramid. The 2 pyramids (4) are called the END-PYRAMIDS.

The vertex-edge and the sides of the base are opposite-edges of a set of tetrahedrons. These tetrahedrons are in a definite cyclical-order corresponding to the sides of the polygon, and are joined each to the next, by the faces which are adjacent to the vertex-edge. They are joined to the end-pyramids by the faces which are adjacent to the sides of the base. The interiors of these tetrahedrons are the LATERAL-CELLS, and these and the interiors of the end-pyramids are the CELLS of the double-pyramid. The pentahedroid is the simplest double-pyramid.

In Fig. 1, we have the graphic-construction of a double-pyramid CO'-OAB. The interior of the segment of the 2 given points C and O' is called the vertex-edge of the double-pyramid, and the interior of the polygon OAB is the base. The interiors of the triangles formed by taking each point of the polygon OAB with the 2 given points C and O' are the elements. The lateral-faces of the double-pyramid are those plane-elements which contain a vertex of the polygon, i.e. the faces CO'O, CO'A, and CO'B are the lateral-faces of the double-pyramid. The 2 end-pyramids are C-OAB and O'-OAB.

The vertex-edge CO' and the sides OA, AB, and BO of the base OAB are opposite-edges of a set of tetrahedrons, i.e. the tetrahedrons OACO', OBCO', and ABCO'. These tetrahedrons are in a definite cyclical-order corresponding to the sides of the polygon OAB, and are joined each to the next by the faces which are adjacent to the vertex-edge; for example, the 2 tetrahedrons OACO' and OBCO' are joined by the face OCO'. These tetrahedrons are joined to the end-pyramids by the faces which are adjacent to the sides of the base OAB; for example, the face OAC is adjacent to the side of the base OAB, and

the end-pyramid C-OAB and tetrahedron OACO' are joined by this face; the face OAO' is also adjacent to the side of the base, and the end-pyramid OABO' is joined to the tetrahedron OACO' by this face; like results follow for the other 2 tetrahedrons and the 2 end-pyramids.

The lateral-cells are OACO', OBCO', and ABCO', and these and the interiors of the end-pyramids are the cells of the double-pyramid.

The intersection of double-pyramids by planes and hyperplanes are given in the following theorems:

Theorem 1. A plane containing a point of the vertex-edge and intersecting the base in the interior of a segment, or a plane containing the vertex-edge and a point of the base, will intersect the double-pyramid in a triangle. (Fig. 6a. and Fig. 6b.)

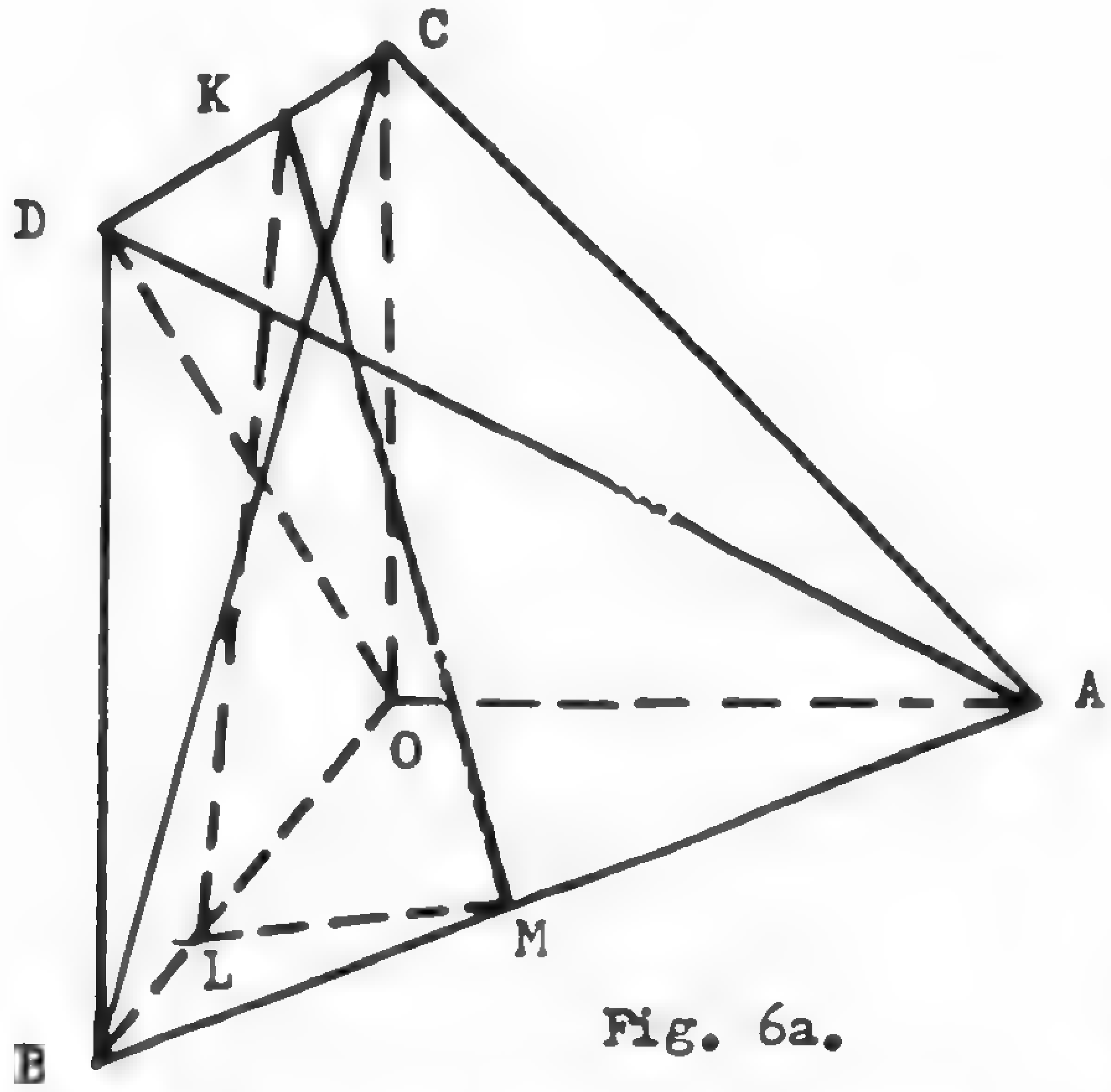


Fig. 6a.

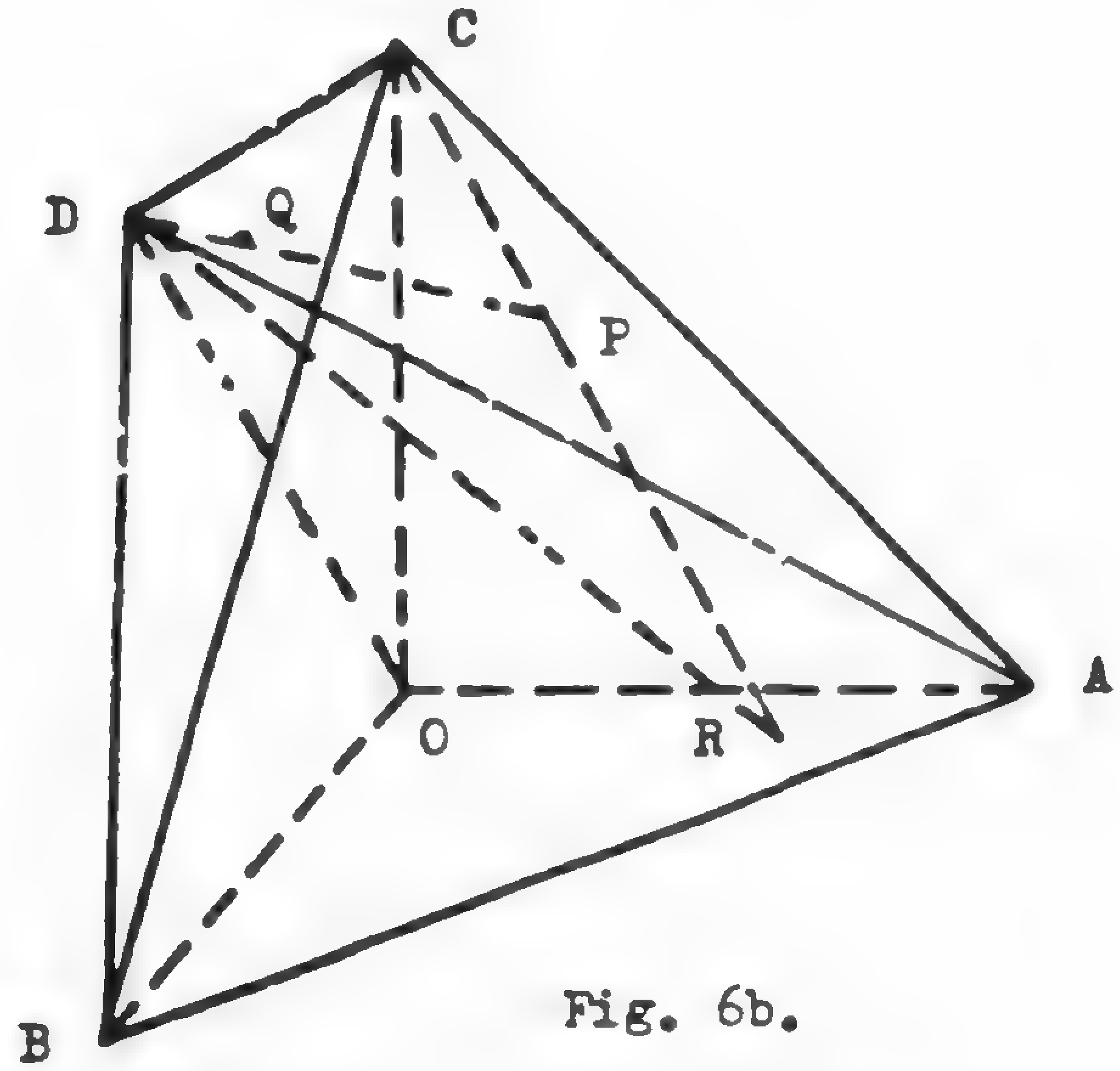


Fig. 6b.

In Fig. 6a, take a plane passing through a point K of the vertex-edge CD and intersecting the base OAB in the interior of a segment LM, then the plane KLM will intersect the double-pyramid CD-OAB in the triangle KLM.

In Fig. 6b, take a plane containing the vertex-edge CD and a point R of the base OAB, then the plane CDR will intersect the double-pyramid CD-OAB in the triangle CDR. In this case, the 2 sides CR and DR of the triangle CDR are in the interiors of the end-pyramids, i.e. CR in C-OAB and DR in D-OAB. The interior of the triangle CDR lies entirely in the interior of the double-pyramid—likewise for the triangle KLM in Fig. 6a.

Theorem 2. A hyperplane containing the base and a point of the vertex-edge will intersect the double-pyramid in a pyramid. (Fig. 7)

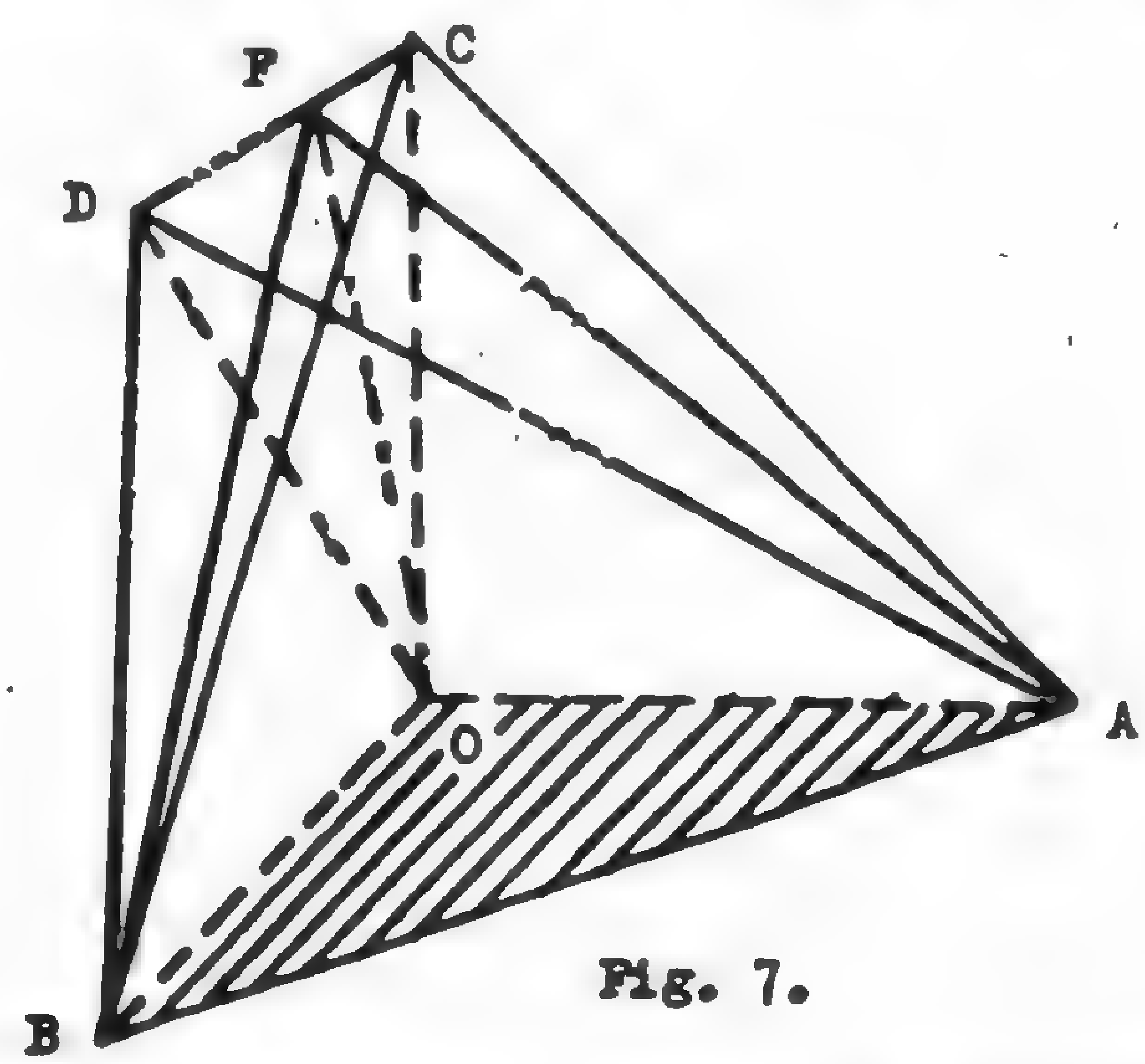


Fig. 7.

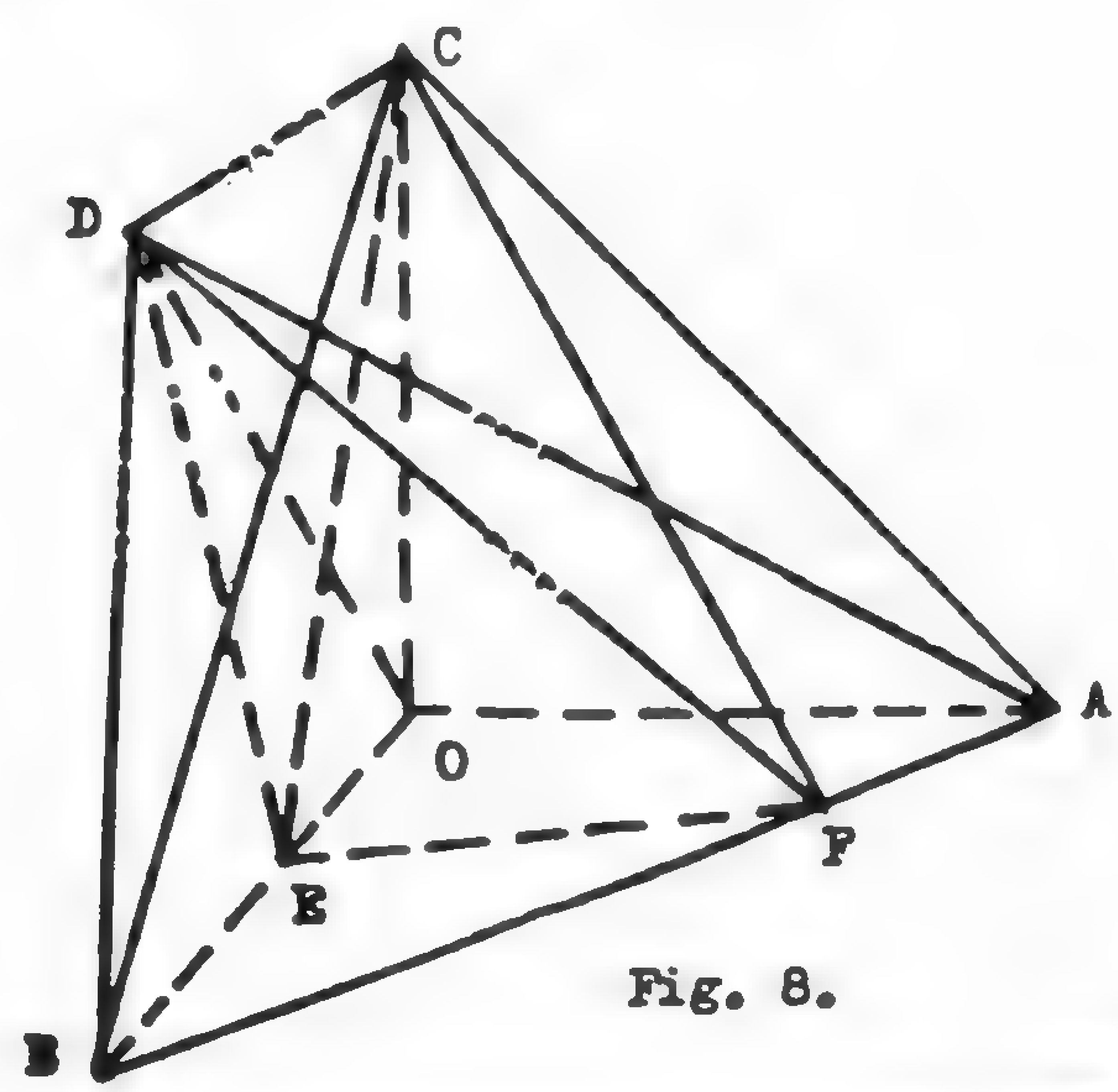


Fig. 8.

In Fig. 7, take a hyperplane containing the base OAB and a point F of the vertex-edge CD, then the hyperplane FOAB will intersect the double-pyramid CD-OAB in the pyramid F-OAB. The interior of the pyramid F-OAB lies entirely in the interior of the double-pyramid CD-OAB.

Theorem 3. A hyperplane containing the vertex-edge and intersecting the base will intersect the double-pyramid in a pyramid. (Fig. 8.)

In Fig. 8, take a hyperplane containing the vertex-edge CD and intersecting the base OAB in the interior of a segment EF, then the hyperplane CDEF will intersect the double-pyramid CD-OAB in the tetrahedron CDEF. The 2 faces CEF and DEF of the tetrahedron CDEF are sections of the interiors of the end-pyramids C-OAB and D-OAB, respectively. The interior of the tetrahedron CDEF lies entirely in the interior of the double-pyramid CD-OAB.

The student will find it to his advantage <u>to study thoroughly</u> this section on the hyperpyramids—compare the <u>pictorial-relationships</u> between the 3- and 4-space graphic-forms, that is, between the single- and double-pyramids.

We shall take up one more important point of the 4-space graphics pertaining to the visualization-process. In the graphic-forms of the hypersolid-geometry, you must see the <u>perceptio-differences</u> when the hyperplane of a cell is <u>viewed</u> from <u>outside</u> of its hyperplane.

For example, if we delete from the pentahedroid OABCD, the red-tetrahedral-cell D-ABC and the interior of the pentahedroid, then we shall see 4 visible-cells in the cut-away-view of the pentahedroid, i.e. the cells OACD, OBCD, OABC, and OABD; the faces, edges, and vertices of these 4 cells are also visible-views.

Suppose we were <u>in</u> the hyperplane of the black-tetrahedron, and deleted the face ABC and the interior of the tetrahedron, then we would see 3 visible-faces in the cut-away-view of the tetrahedron, i.e. the faces OAC, OBC, and OAB; the edges and vertices of these 3 faces are also visible-views.

9. HYPERCONICAL-HYPERSURFACES—<u>Undefined-Terms</u>. HYPERCONES—<u>Definitions</u>.

HYPERSURFACE is the term applied to a figure in hyperspace which corresponds to the surfaces of geometry of 3 dimensions. The term 'hypersurface' is left undefined. We Shall use the word only in connection with certain simple-figures which we shall define individually. The hyperplane is the simplest hypersurface.

A HYPERCONICAL-HYPERSURFACE consists of the lines determined by the points of a hyperplane-surface and a point not in the hyperplane of this surface. The point is the VERTEX, the surface is the DIRECTING-SURFACE, and the lines are the ELEMENTS. The hyperconical-hypersurface has 2 nappes.

The only hyperconical-hypersurfaces which we shall consider at present are those in which the directing-surface is a plane, a sphere, a circular-conical-surface, or a part or combination of parts of such surfaces. When the directing-surface is a plane, the hypersurface is a hyperplane or a portion of a hyperplane.

A HYPERCONE consists of a hyperplane-surface, or portions of hyperplane-surfaces, forming a closed-hyperplane-figure, and a point not a point of the hyperplane of this figure, together with the interior of the latter and the interiors of the segments formed by taking the given point with the points of the hyperplane-figure. The point is the VERTEX, the interiors of the segments are ELEMENTS, and the interior of the hyperplane-figure is the base.

The hyperpyramid may be reguarded as a particular-case of the hypercone. The only other cases which we shall consider at present are those in which the base is the interior of a sphere or of a circular-cone.

A plane containing the vertex of a hypercone and intersecting the base in the interior of a segment, will intersect the hypercone in a triangle; and a hyperplane containing the vertex and intersecting the base, will intersect the hypercone in a cone.

The INTERIOR OF A HYPERCONE consists of the interiors of the segments formed by taking the vertex with the points of the base, but in the case of a convex-hypercone the interior of any segment whose points are points of the hypercone will lie entirely in the interior of the hypercone unless it lies entirely in the hypercone itself. No line can intersect a convex-hypercone in more than 2 points if it passes through a point of the interior, and any half-line drawn from a point O of the interior will intersect the hypercone in 1 and only 1 point.

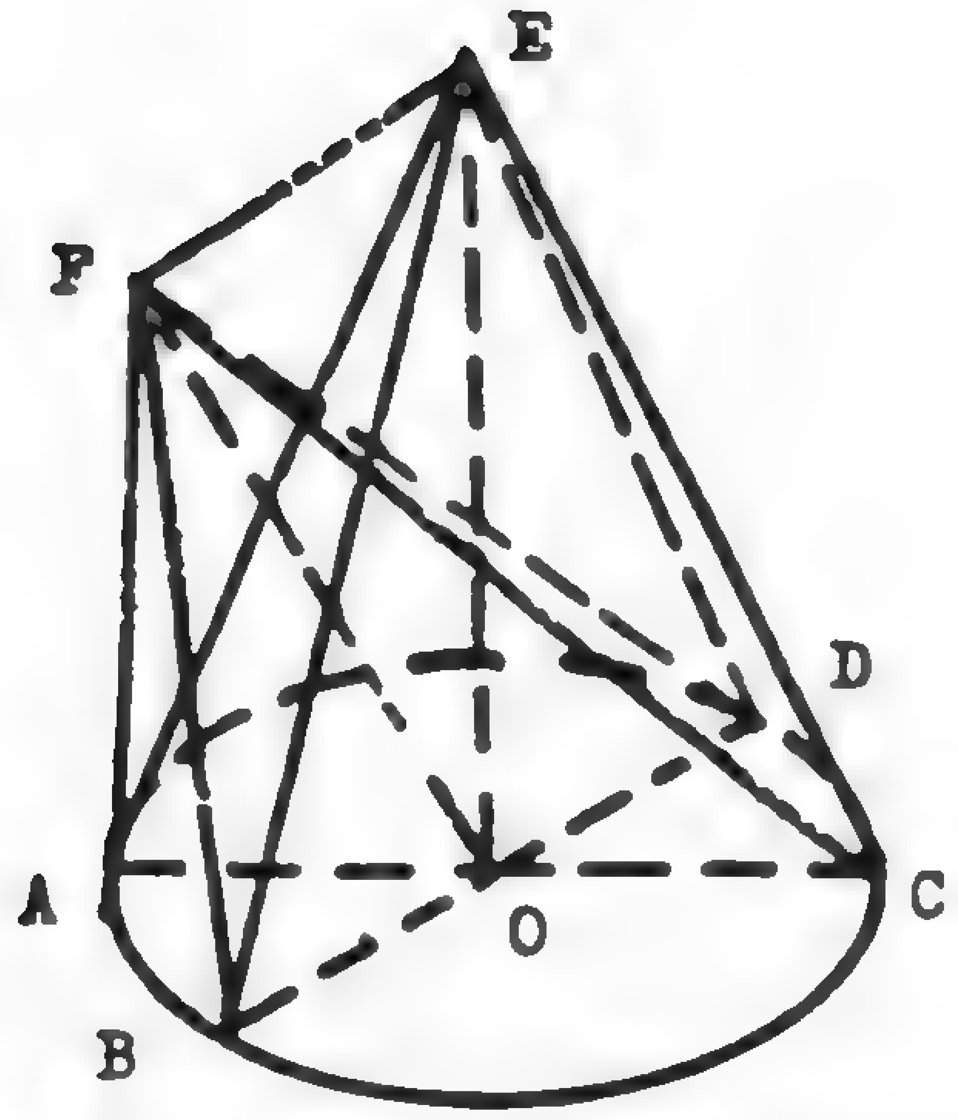


Fig. 9.

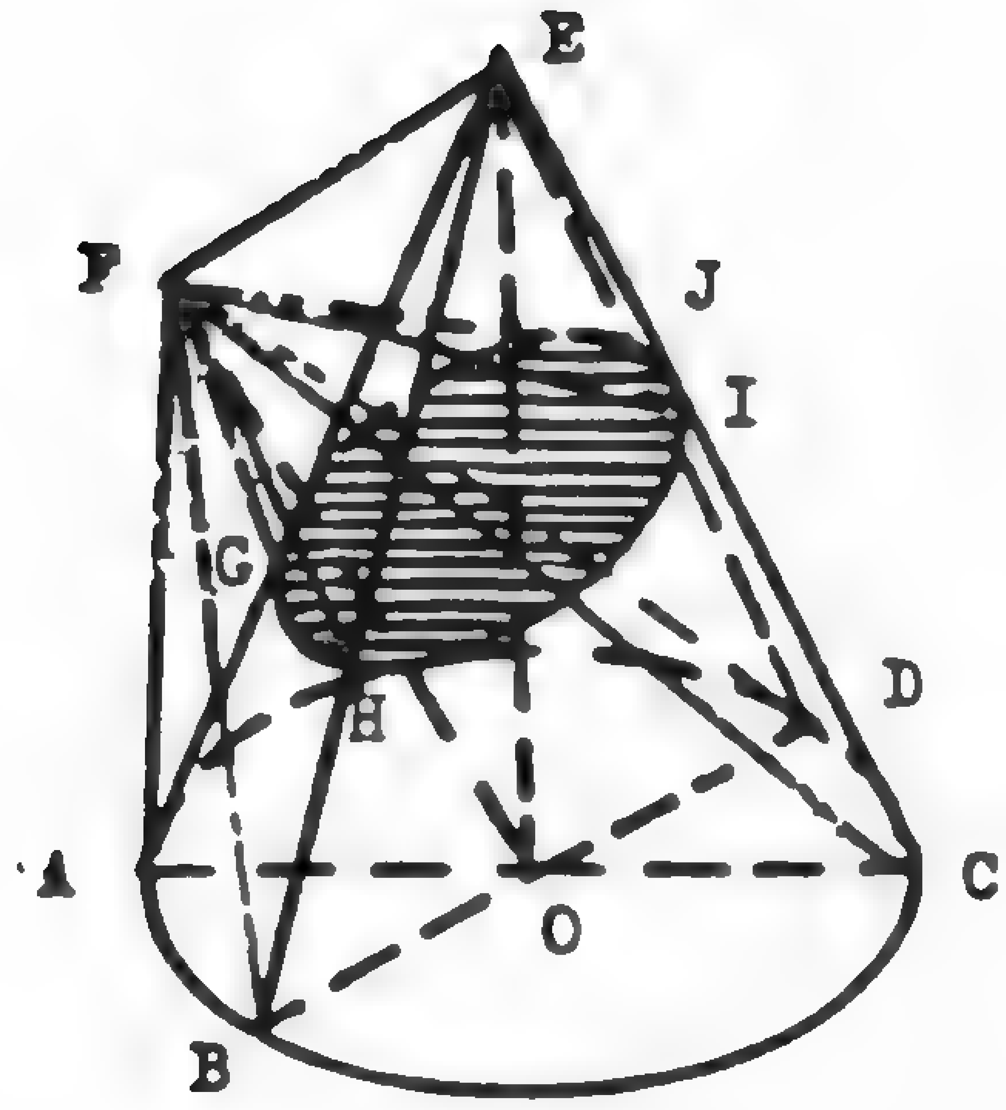


Fig. 10.

In Fig. 9, we have the graphic-construction of a hypercone. We shall designate the hypercone as F-E(ABC). The 3 capital-letters ABC enclosed in parentheses shall designate a 'closed-plane-curve', and a bar placed over the 3 capital-letters shall designate an 'arc of a closed-plane-curve'.

Take a closed-hyperplane-figure such as a cone E-(ABC) and a point F not in the hyperplane of this cone, together with the interior of this cone and the interiors of all segments formed by taking the point F with the points of the given hyperplane-cone. The point F is called the vertex, the interiors of the segments FE, FA, FB, FC,... are called the elements of the hypercone F-E(ABC), and the interiors of the hyperplane-cone E-(ABC) is the base.

The interior of the hypercone F-E(ABC) consists of the interiors of the segments formed by taking the vertex F with the points of the base E-(ABC).

The visible- and hidden-views of the hypercone correspond to the ordinary-cone of the 3-space solid-geometry.

In another chapter we shall take up the study of hyperspace-rotations, that is, a rotation around a plane in hyperspace, and a rotation around a plane lying in a given hyperplane—this information on hyperspace-rotations makes it possible for us to represent visible- and hidden-views of curved-hypersurfaces, the visible- and hidden-views of closed-curved-hypersurfaces will be taken up later.

In the hypercone F-E(ABC), we then have the following visible- and hidden-views: ½-hypersurface of the ½-hypercone F-E($\overline{ABC}$) will be a visible-view in the graphic-drawing; the other ½-hypersurface of the hypercone F-E(ABC) will be the ½-hypercone F-E($\overline{CDA}$) which represents a hidden-view in the graphic-drawing.

A hypercone, or the hypersolid which we call the interior of the hypercone, can be somewhat described as cut from 1 nappe of a hyperconical-hypersurface by the hyperplane of the directing-surface.

Theorem 1. A hyperplane containing the vertex and intersecting the base will intersect the hypercone in a cone. (Fig. 10.)

In Fig. 10, take a hyperplane containing the vertex F and intersecting the base E-(ABC) in the plane of a closed-curve (GHI), then the hyperplane FGHI will intersect the hypercone F-E(ABC) in a cone F-(GHI).

Theorem 2. A hyperplane passing between the vertex of a hypercone and the base will intersect the hypercone in a cone. (Fig. 11.)

In Fig. 11, take a hyperplane PKLM passing between the vertex F of a hypercone F-E(ABC) and the base E-(ABC), then the hyperplane PKLM will intersect the hypercone F-E(ABC) in a cone P-(KLM).

9. DOUBLE-CONES—Definitions. A hypercone whose base is the interior of a cone may be reguarded in 2-ways as a hypercone of this kind, the vertex of the base in one-case being the vertex of the hypercone in the other-case.

Thus their are 2 cones having themselves a common-base, and we can say that the hypercone is determined by a closed-plane-curve and 2 points neither of which is in the hyperplane containing the curve and the other point. Looked at in this way the hypercone

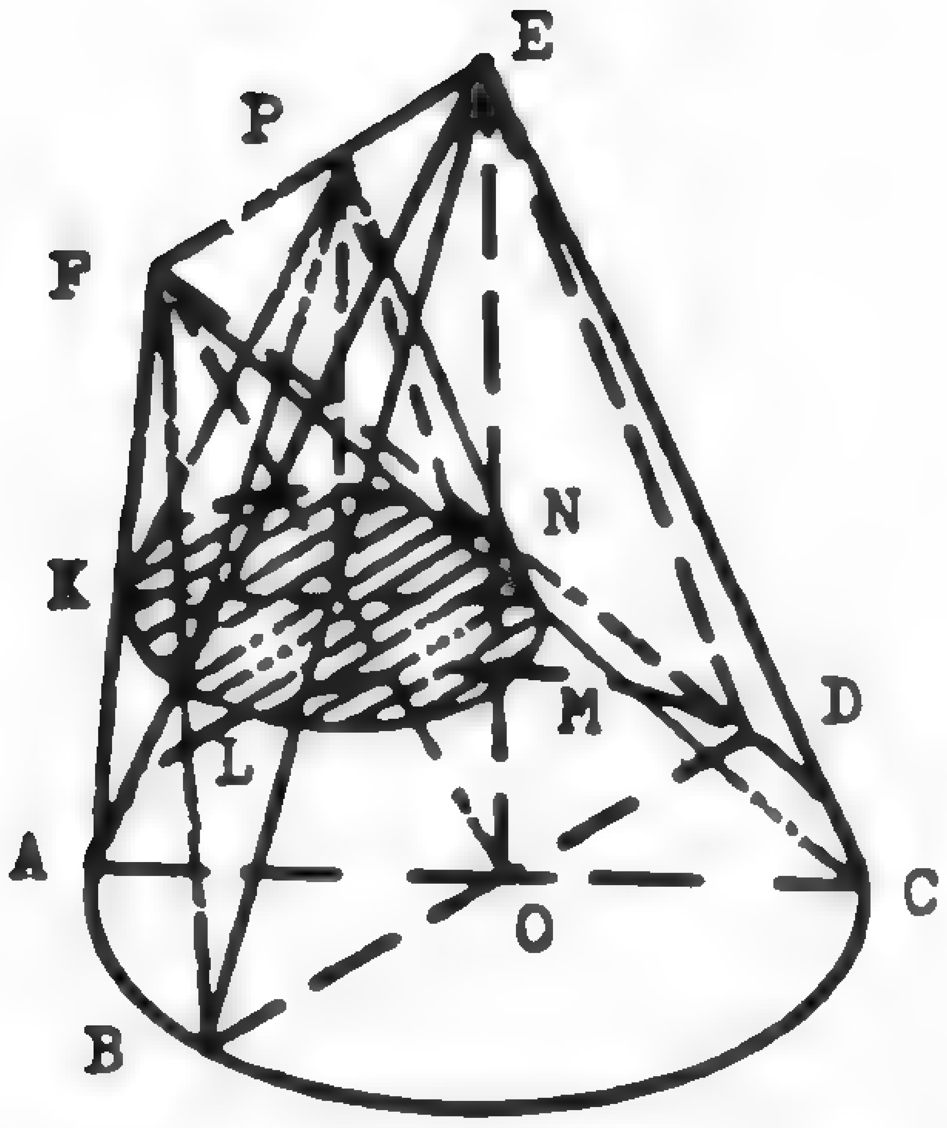


Fig. 11.

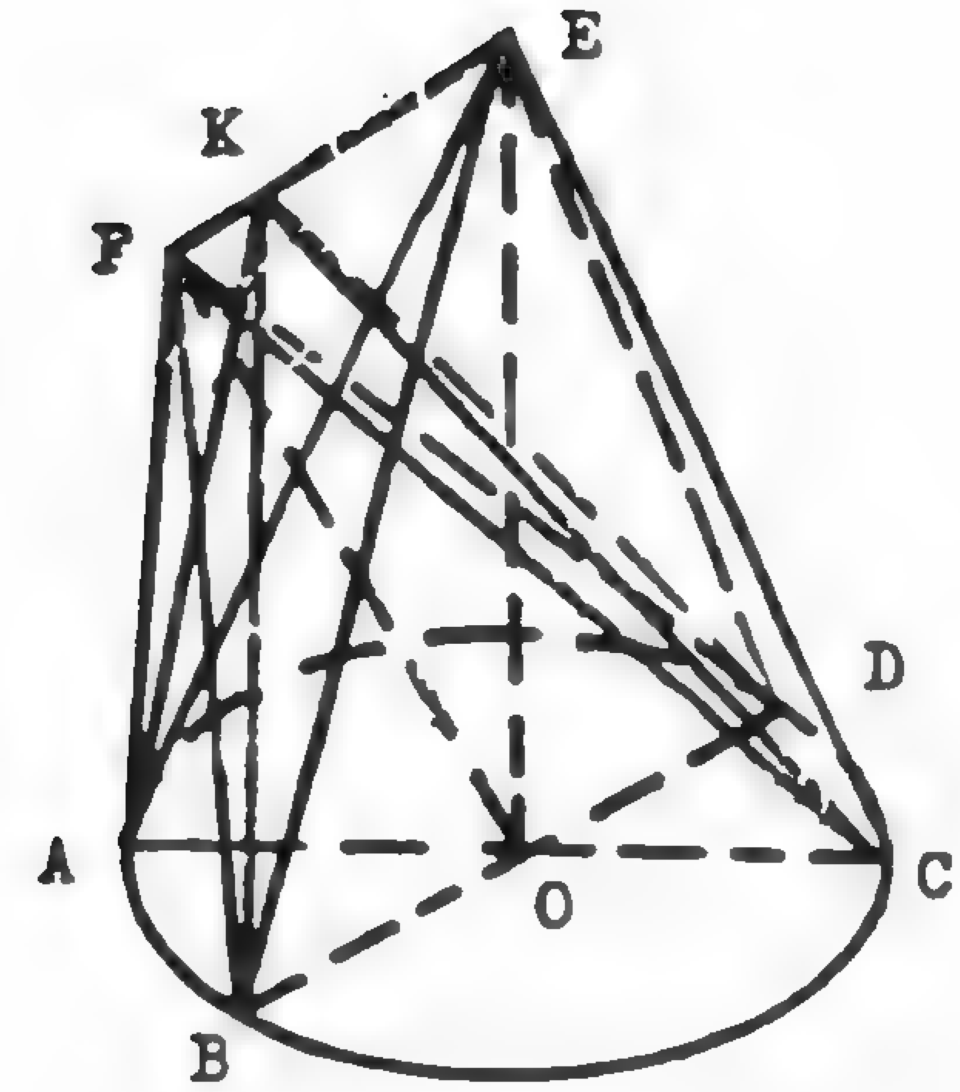


Fig. 12.

is called a DOUBLE-CONE.

A DOUBLE-CONE consists of the following classes of points:

(1) the points of a closed-plane-curve and the points of its interior;

(2) 2 points not in a hyperplane with the curve, the interior of the segment formed of these 2 points, and the interiors of the segments formed by taking each of these points with the points of the curve;

(3) the interiors of the triangles formed by taking each point of the curve with the 2 given points;

(4) the interiors of 2 cones each formed by taking the interior of the closed-plane-curve with 1 of the 2 given points.

The interior of the segment of the 2 given points is the VERTEX-EDGE of the double-cone; and the interior of the curve is the BASE. The interiors of the triangles (3) are the ELEMENTS, and the 2 cones (4) are the END-CONES.

Certain cases of intersection of double-cones are given by the following theorems:

Theorem 1. A hyperplane containing the base and a point of the vertex-edge will intersect the double-cone in a cone. (Fig. 12.)

In Fig. 12, take a hyperplane containing the base (ABC) and a point K of the vertex-edge EF, then the hyperplane KABC will intersect the double-cone EF-(ABC) in the cone K-(ABC).

Theorem 2. A hyperplane containing the vertex-edge and intersecting the base will intersect the double-cone in a tetrahedron. (Fig. 9.)

In Fig. 9, take a hyperplane containing the vertex-edge EF and intersecting the base (ABC) in the interior of a segment BD, then the hyperplane EFBD will intersect the double-cone EF-(ABC) in the tetrahedron EFBD. This tetrahedron will have 2 faces EBD and FBD lying in the end-cones E-(ABC) and F-(ABC), respectively; the other 2 faces EFB and EFD of this thetrahedron will lie in the hyperconical-hypersurface of the double-cone—the interior of this tetrahedron lies entirely in the interior of the double-cone.

Theorem 3. A plane containing a point of the vertex-edge and intersecting the base in the interior of a segment, or a plane containing the vertex-edge and a point of the base , will intersect the double-cone in a triangle. (Fig. 12.)

In Fig. 12, take a plane containing a point K of the vertex-edge EF and intersecting the base (ABC) in the interior of a segment BD, then the plane KBD will intersect the double-cone EF-(ABC) in the triangle KBD; and a plane containing the vertex-edge EF and a point O of the base (ABC) will intersect the double-cone EF-(ABC) in the triangle EFO. The interiors of the triangles KBD and EFO lie entirely in the interior of the double-cone EF-(ABC).

10. PLANO-CONICAL-HYPERSURFACES—Definitions. A PLANO-CONICAL-HYPERSURFACE consists of the planes determined by the points of a plane-curve and a line not in the hyperplane with the curve.

The line is the VERTEX-EDGE, the curve is the DIRECTING-CURVE, and the planes are the ELEMENTS. Each element meets the plane of the directing-curve in only 1 point, the point where it meets the directing-curve itself. Their are 2 nappes to the hypersurface.

For the present, we shall consider only the case in which the directing-curve is a circle.

Theorem. A hyperplane which contains the directing-curve of a plano-conical-hypersurface and a point of the vertex-edge intersects the hypersurface in a conical-surface.

The line containing the vertex-edge of a double-cone, and the curve whose interior is the base, are the vertex-edge and directing-curve of a plano-conical-hypersurface. (See Fig. 9, the hypersurface of a double-cone EF-(ABC) is a restricted-portion of a plano-conical-hypersurface.)

A double-cone, or the hypersolid which we call the interior of a double-cone, can be somewhat described as cut from 1 nappe of a plano-conical-hypersurface by 2 hyperplanes each of which contains the directing-curve and a point of the vertex-edge. (See Fig. 9, let the cutting-hyperplanes be that of the hyperplanes of the end-cones E-(ABC) and F-(ABC).

1-13-83

Dear Fry:

To visualize 9-magnetics as applied to hyperspace-jumps in power-multiples of c, take a <u>log-graph</u> and plot these functions:

$$y_i = c^{2^{x_i}}, \text{ where } x_i = i(i = 0,1,2,\ldots)$$

$$t_{y_i} = x_i + t \quad : \text{using the log-graph } \underline{\text{twice}}, \text{ doubly.}$$

Example: Take $i = 1$, then $y_1 = c^{2^{x_1}} = c^{2^1} = c^2$, and $t_{y_1} = x_1 + t = 1 + t$(future-time-frame): we have also $t_{c^2} = 1 + t$. Take $i = 2$, then $y_2 = c^4$, $t_{y_2} = 2 + t$, or $t_{c^4} = 2 + t$.

On the log-graph $y = y_i$, $x = x_i$.

This is a crude-approximation only, nevertheless quite accurate otherwise...

Using future-time-frames corresponding to the Planck-Constant power-multiples, which in turn correspond to power-multiples of c, we can compare the future-time-frames that <u>successively</u> <u>overlap</u> in 1 second time-intervals as follows:

$$\begin{array}{llllll} c & c^2 & c^4 & c^8 \ldots & c^{2^{x_i}} \\ t & x_1+t & x_2+t & \ldots & x_i+t \\ t & 1+t & 2+t & 3+t \ldots & i+t \text{ (} i \text{ in seconds) .} \end{array}$$

In effect, what U have is <u>FOURIER STEP-FUNCTIONS</u> of overlapping time-frames, and h^{i+1} represents the Planck-Constant <u>timing-principle</u> in power-multiples of h.

Observe that $(x_{i+1} + t) - (x_i + t) = ((i+1) + t - (i + t) = 1 + t$(an <u>invariant</u> of 1 second <u>difference</u> separating <u>adjacent</u> time-frames that overlap slightly, and the 'higher' quantum time-frame 1 second in the 'future'. Therefore we see that the Planck-Constant is a timing-principle separating adjacent time-frames, but this can only be achieved using $h^{x_i+1}_{hid} = h^{i+1}$. If $i = 0$, we merely have <u>h, c, and t</u> (ground-state).

This info is 'alien' to the defunct Einstein-Physics, since we are dealing with <u>MAGNETIC-FIELDS</u> and <u>INERTIALESS-SYSTEMS</u>.

I do know for sure that the Einstein-physics is a DEAD-physics that has little relevancy to starship travel.

As Einstein said, "Nothing can move faster than · (light)". Ha! Einstein-Physics is a <u>Frozen-Frame</u> physics only, period.

Sincerely Yours,

George L. Brandes

1200 E. Sherman Apt 1
Cd'A, ID 83814

<u>Ground-State</u> h,c,t
NOW Main-Line-Time

Analogy: Like a stack of overlapping film-strips...

CHAPTER III

I. LINES PERPENDICULAR TO A HYPERPLANE

11. EXISTENCE OF PERPENDICULAR LINES AND HYPERPLANES.

Notation: The symbol for perpendicular is $\perp$; for perpendiculars, $\perp$s .

Theorem 1. The lines perpendicular to a line at a given point do not all lie in 1 plane.

Proof: Every point in hyperspace lies in a plane with the given line, and in every plane which contains the line there is a $\perp$ to the line at the given point. Now if these $\perp$s were all in 1 plane, that plane and the given line would determine a hyperplane containing all the planes which contain the line (Art. 1, Th. 1), and so all points of hyperspace. But the points of hyperspace do not all lie in 1 hyperplane.

Theorem 2. A line perpendicular at a point to each of 3 non-coplanar lines, is perpendicular to every line through this point in the hyperplane which the 3 lines determine. (Fig. 13.)

Given: A line m $\perp$ at a point O to each of 3 non-coplanar lines a, b, and c.

To Prove: The line m is $\perp$ to every line through O in the hyperplane which the 3 lines a, b, and c determine.

Proof: Let d be any other line through O in the hyperplane determined by these 3 lines. The plane of cd will intersect the plane of ab in a line h (RSG-III, Th. 1). The line m, being $\perp$ to a and b, is $\perp$ to h lying in the plane of ab; and then, being $\perp$ to c and h, it is $\perp$ to d lying in the plane of ch.* Therefore the line m is $\perp$ to every line through O in the hyperplane which the 3 lines a, b, and c determine. (Q.E.D)

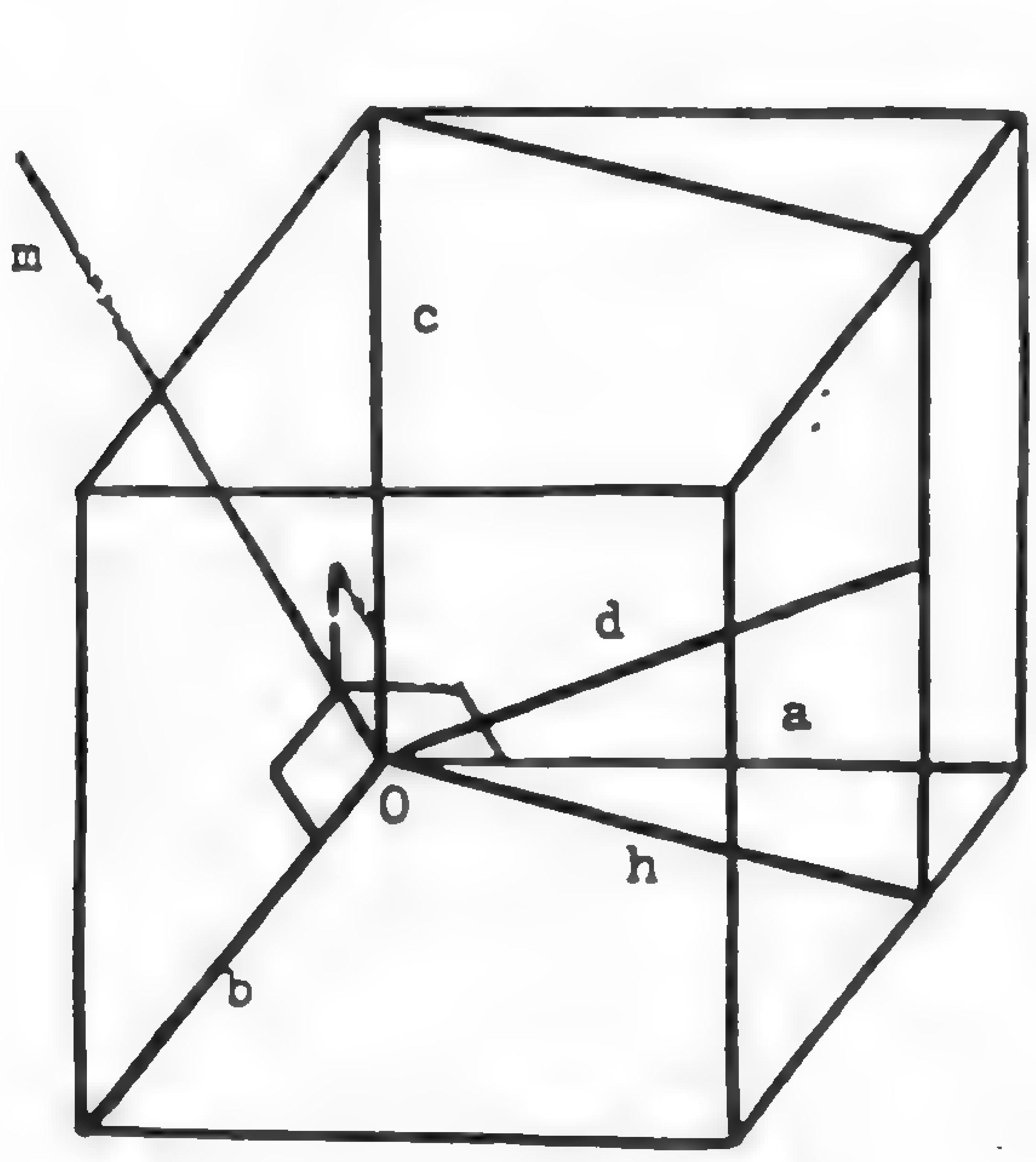


Fig. 13.

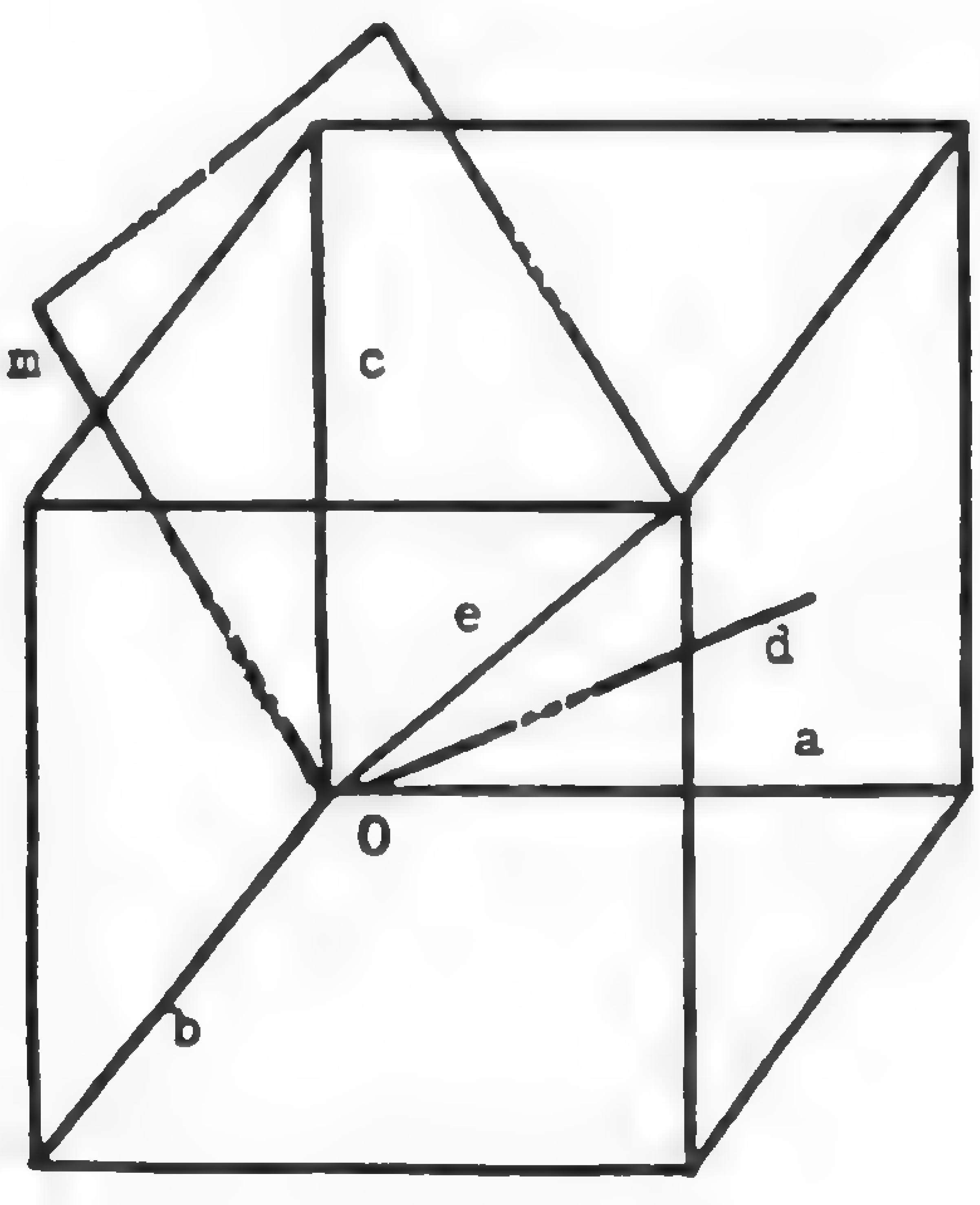


Fig. 14.

* A line m through a point O, $\perp$ to each of 2 lines intersecting at O, is $\perp$ to every line through O in the plane which the 2 lines determine. This is always true, for the plane and the line m lie in 1 hyperplane.

Theorem 3. All lines $\perp$ to a given line at a given point lie in 1 hyperplane. (Fig. 14.)

Given: A line m and a point O of m.

To Prove: All lines $\perp$ to the line m at a point O lie in 1 hyperplane.

Proof: 3 non-coplanar lines a, b, and c $\perp$ to m at O determine a hyperplane of abc such that m is $\perp$ to every line in the hyperplane of abc through O (Th. 2). Now let d be any other line $\perp$ to m at O. The plane of dm intersects the hyperplane of abc in a

line e (Art. 4, Th. 1), also $\perp$ to m at O. In the plane of dm, then, we have a line m and the 2 lines d and e $\perp$ to m at O. For this reason d must coincide with e and lie in the hyperplane of abc. Therefore all lines $\perp$ to the line m at a point O lie in 1 hyperplane. (Q.E.D)

A line intersecting a hyperplane at a point O is PERPENDICULAR TO THE HYPERPLANE when it is perpendicular to all lines of the hyperplane which pass through O; the hyperplane is also said to be PERPENDICULAR TO THE LINE. The point O is called the FOOT of the perpendicular.

12. 1 HYPERPLANE THROUGH ANY POINT PERPENDICULAR TO A GIVEN LINE. PLANES IN A PERPENDICULAR-HYPERPLANE.

Theorem 1. At any point of a line there is 1 and only 1 hyperplane perpendicular to the line.

This follows from the theorems of the preceding article.

Theorem 2. Through any point outside of a line passes 1 and only 1 hyperplane perpendicular to the line.

Theorem 3. A line perpendicular to a hyperplane is perpendicular to every plane of the hyperplane passing through the foot of the line; and every plane perpendicular to a line at a point lies in the hyperplane perpendicular to the line at this point.

Theorem 4. If a line and a plane intersect, a line perpendicular to both at their point of intersection is perpendicular to the hyperplane determined by them; or if 2 planes intersect in a line, a line perpendicular to both at any point of their intersection is perpendicular to the hyperplane determined by them.

13. LINES PERPENDICULAR TO A GIVEN HYPERPLANE.

Theorem 1. At a given point in a hyperplane there is 1 and only 1 line perpendicular to the hyperplane. (Fig. 15.)

Given: A hyperplane OABC and a point O in the hyperplane.

To Prove: A line m $\perp$ to the hyperplane OABC at the point O, is the only line $\perp$ to the hyperplane.

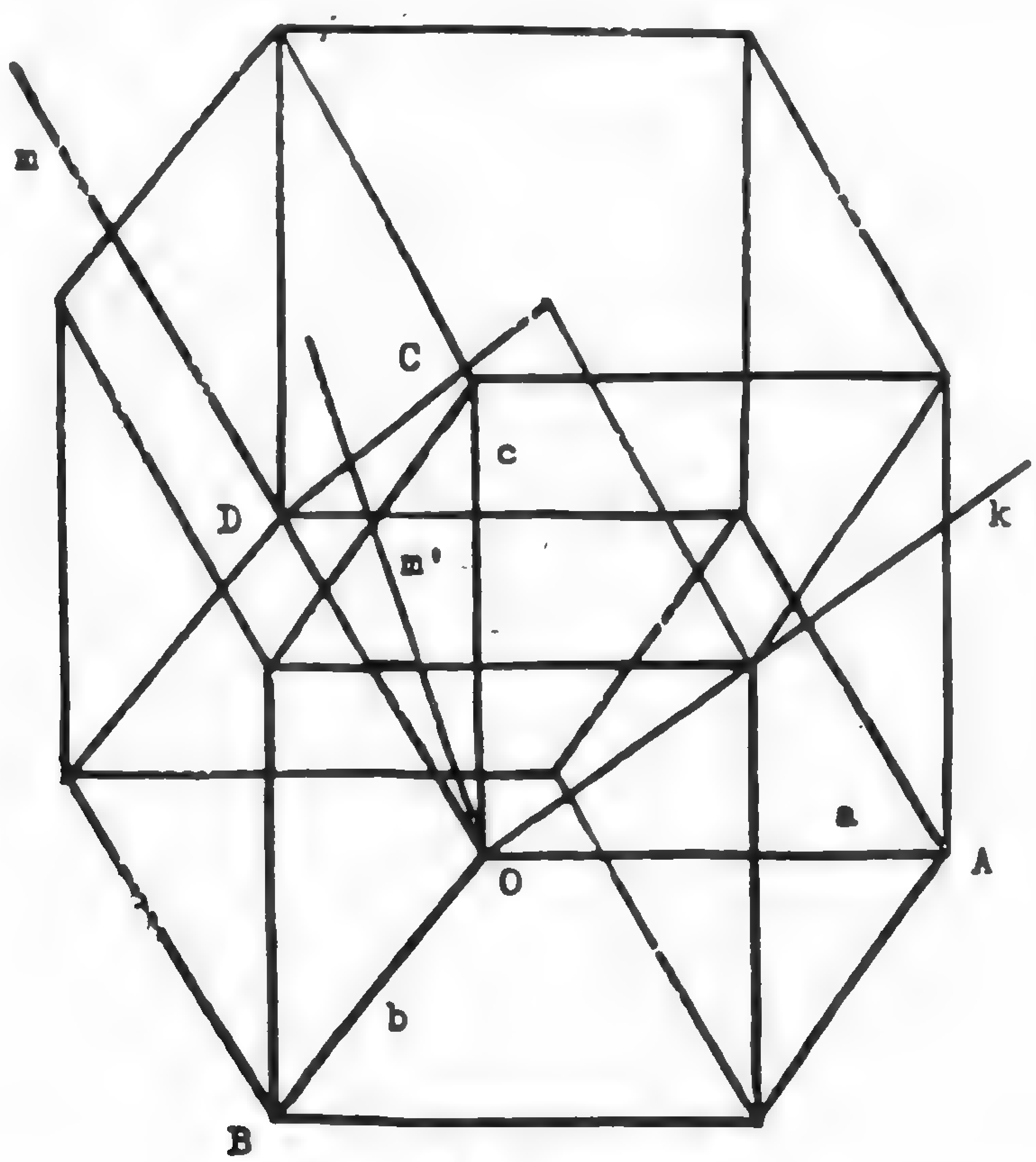


Fig. 15.

Proof: Consider 3 non-coplanar lines a, b, and c lying in the hyperplane OABC and passing through the point O. The 3 hyperplanes OABD, OBCD, and OCAD $\perp$ to a, b, and c at the point O, respectively, have in common at least a line m (Art 4, Th. 2 and remark), and any such line m must be $\perp$ to the hyperplane OABC because $\perp$ to the 3 lines a, b, and c.

If there were 2 lines m and m' $\perp$ to the hyperplane OABC at the point O, they would both be $\perp$ to a line k in which their plane of mm' intersects the hyperplane OABC. We should have in the plane of mm' 2 lines m and m' $\perp$ to a 3rd line k at the same point O, which is impossible. For this reason m' coincides with m. Therefore a line m $\perp$ to the hyperplane OABC at the point O, is the only line $\perp$ to the hyperplane. (Q.E.D)

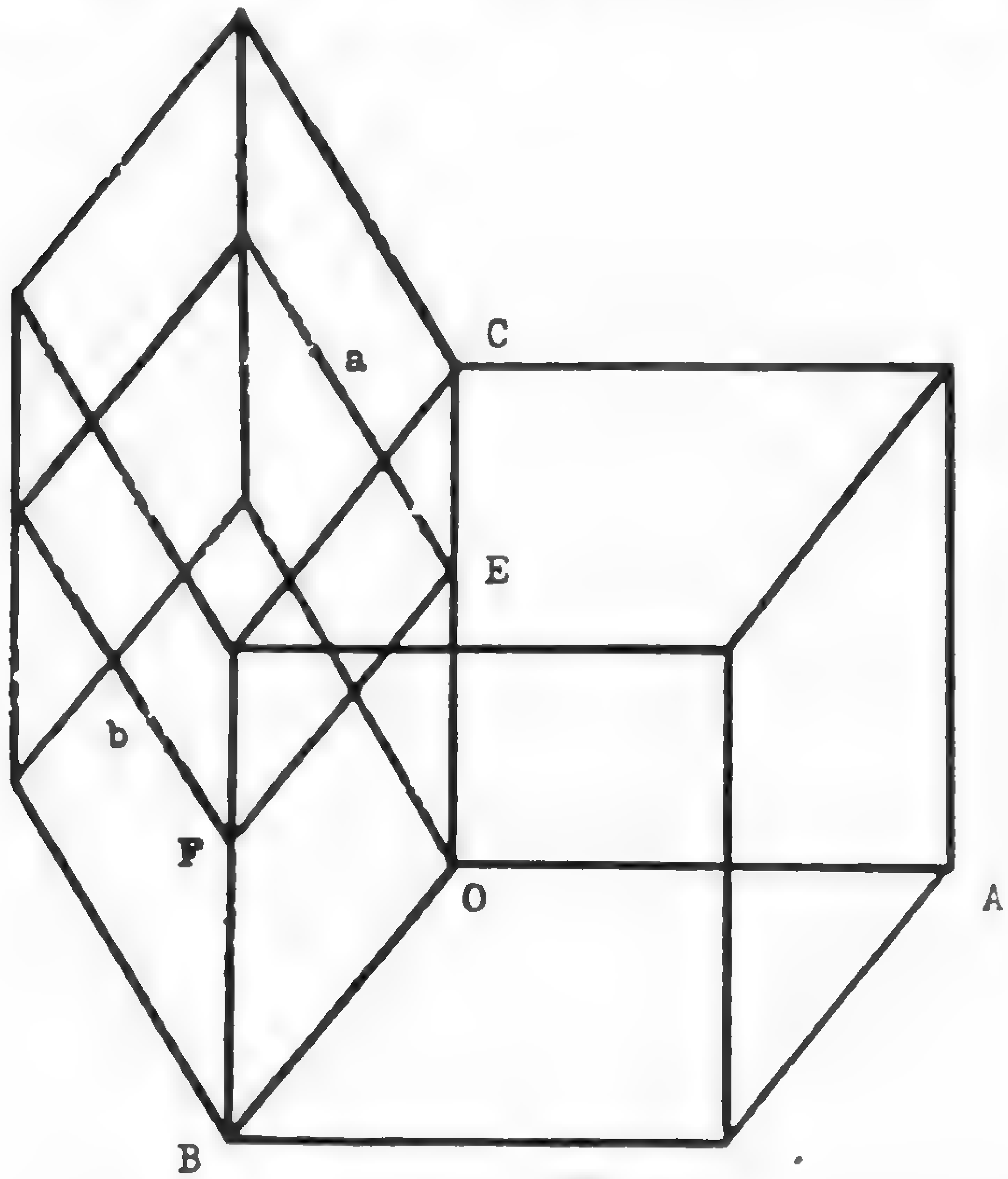


Fig. 16.

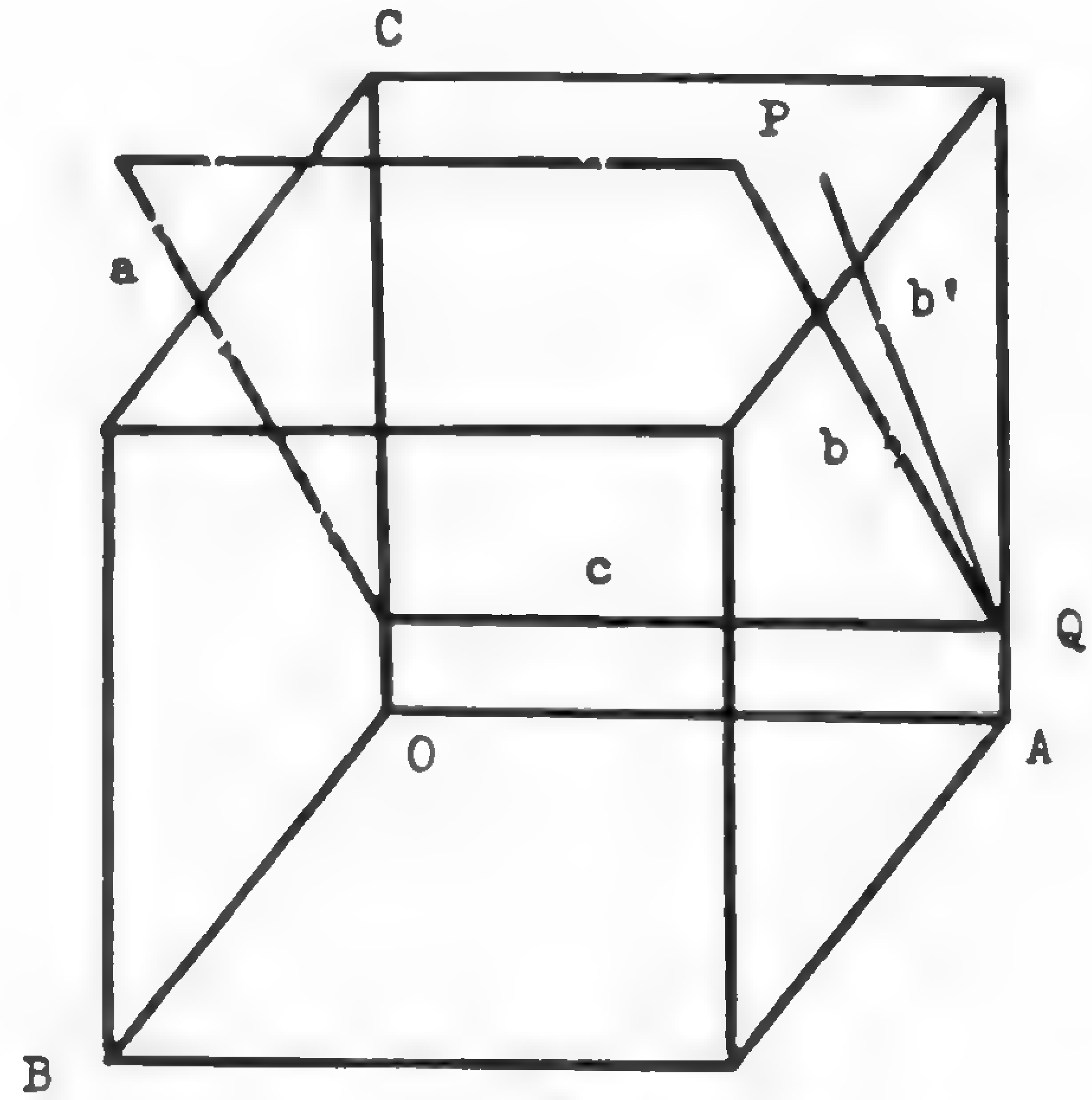


Fig. 17.

Theorem 2. 2 lines perpendicular to a hyperplane lie in a plane. (Fig. 16.)

Given: 2 lines a and b $\perp$ to a hyperplane OABC at points E and F respectively.

To Prove: The 2 lines a and b lie in the plane of aF.

Proof: Construct a hyperplane containing the 2 lines a and b. Now any 2 lines lie in a hyperplane (Art. 1, Th. 2 (2)), and a hyperplane containing the 2 lines a and b, intersects the hyperplane OABC in a plane to which the lines a and b are both $\perp$ (Art. 12, Th. 3). For this reason as proved in the solid-geometry, the lines a and b lie in the plane aF. (Q.E.D)

Theorem 3. Through any point outside of a hyperplane passes 1 and only 1 line perpendicular to the hyperplane. (Fig. 17.)

Given: Any point P outside of a hyperplane OABC.

To Prove: 1 and only 1 line passing through P can be $\perp$ to the hyperplane OABC.

Proof: Construct a line a $\perp$ to the hyperplane OABC. If a does not pass through P, a and P determine a plane, the plane of aP intersects the hyperplane OABC in a line c and in the plane of aP there is a line b through P $\perp$ to c, intersecting c at a point Q. Let b' be the line $\perp$ to the hyperplane at Q. a and b' lie in a plane (Th. 2), which is the plane of aQ containing a and the point Q. But the plane determined by a and P passes through Q. Therefore b' lies in the plane determined by a and P, and in the plane

of aP is $\perp$ to c at Q. But we then have, b' coinciding with b, and b must be $\perp$ to the hyperplane OABC.

If there were 2 lines through P $\perp$ to the hyperplane OABC, we should have 2 lines through P $\perp$ to the line which passes through their feet, and this is impossible. Therefore 1 and only 1 line passing through P can be $\perp$ to the hyperplane OABC. (Q.E.D)

13A PROJECTIONS—Definitions. The orthogonal-PROJECTION OF A POINT UPON A HYPERPLANE is the foot of the perpendicular from the point to the hyperplane. The perpendicular can also be called the projecting-line.

Theorem 1.* The distance from any point outside of a hyperplane to its projection upon the hyperplane is less than the distance from the point to any other point of the hyperplane.

The DISTANCE BETWEEN A HYPERPLANE AND A POINT OUTSIDE OF THE HYPERPLANE is the distance between the point and its projection upon the hyperplane.

Corollary. If the distance between 2 points is less than the distance of 1 of them from a hyperplane, they lie on the same-side of the hyperplane in hyperspace.

Theorem 2. Given any point P outside of a hyperplane, and O its projection upon the hyperplane, then any 2 points of the hyperplane equally-distant from P will be equally-distant from O, and any 2 points equally-distant from O will be equally-distant from P; and if 2 points of the hyperplane are unequally-distant from either P or O, that point which is nearer to 1 of them will be nearer to the other. (Fig. 18.)

Given: Any point P outside of a hyperplane OABC, and O its projection upon the hyperplane.

To Prove: Any 2 points A and B of the hyperplane OABC equally-distant from P will be equally-distant from O, and conversely, any 2 points A and B equally-distant from O will be equally-distant from P; and if 2 points A and D of the hyperplane OABC are unequally-distant from either P or O, that point which is nearer to 1 of them will be nearer to the other.

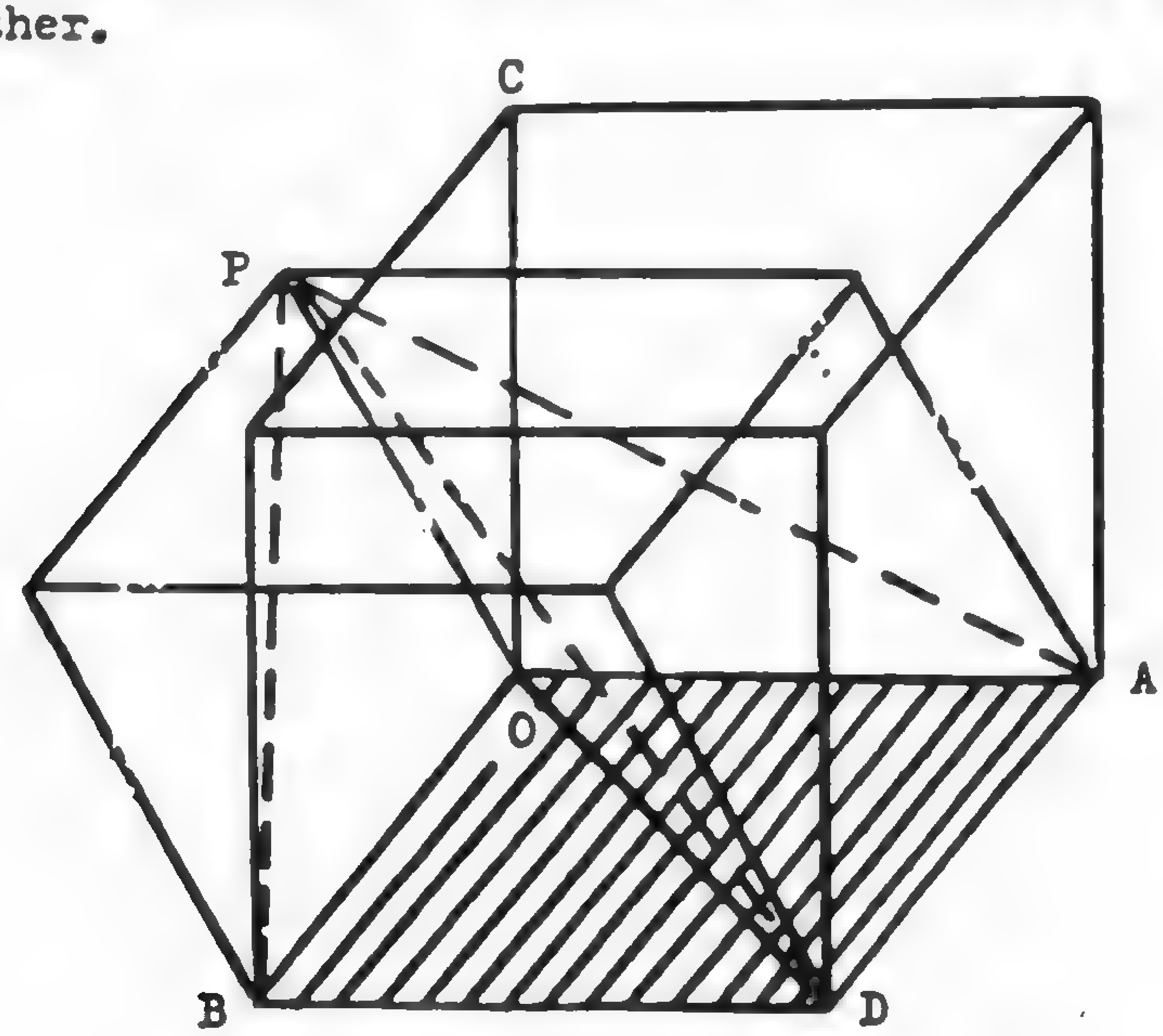


Fig. 18.

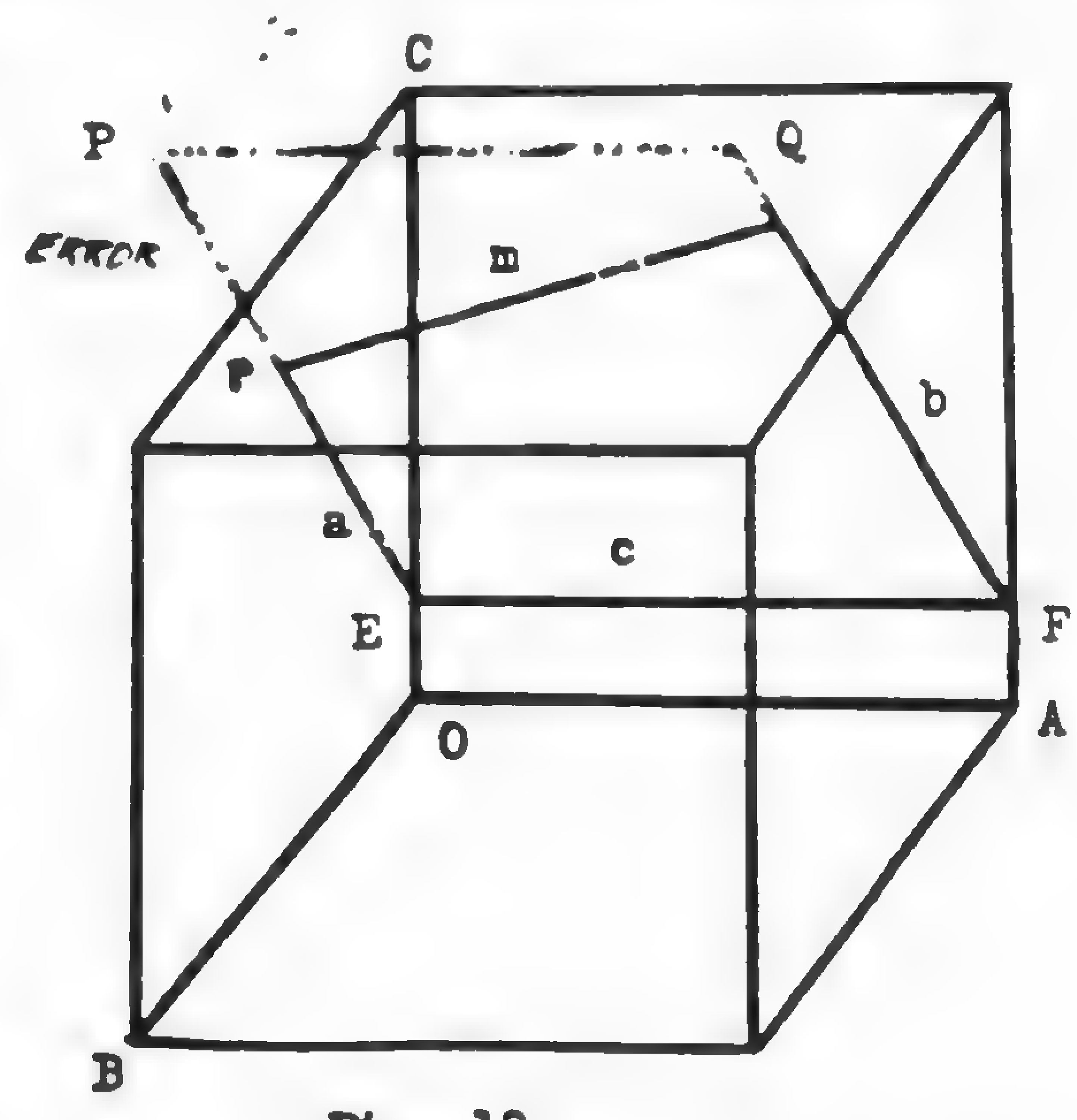


Fig. 19.

Proof: The $\perp$ PO, and the lines through P and any 2 points A and B of the hyperplane OABC, lie in a 2nd hyperplane POAB which intersects the given hyperplane OABC in a plane OAB. The $\perp$ PO is $\perp$ to OAB, and the 3 lines PO, PA, and PB intersect the plane OAB lying in the hyperplane OABC in the points O, A, and B, respectively. The theorem is therefore a theorem in the solid-geometry of the 2nd hyperplane POAB.

In the plane OAB lying in POAB, construct AP = BP, then AO = BO, and conversely, if AO = BO, then AP = BP. In the plane OAB lying in POAB, construct DO greater than AO, then DP is greater than AP, and conversely, if DP is greater than AP, then DO is greater than AO. Therefore the theorem is proved. (Q.E.D)

*The theorems of this article are true at least when the distances referred to are 'restricted'.

14. PROJECTION OF A LINE UPON A HYPERPLANE. ANGLE OF A $\frac{1}{2}$-LINE AND HYPERPLANE. The projection of any figure upon a hyperplane consists of the projections of its points.

Theorem 1. When a line and a hyperplane are not perpendicular, the projection of the line upon the hyperplane is a line or a part of a line. (Fig. 19.)

Corollary. When a $\frac{1}{2}$-line drawn from a point O of a hyperplane does not lie in the hyperplane and is not perpendicular to it, its projection upon the hyperplane is a $\frac{1}{2}$-line drawn in the hyperplane from O, or the interior of a segment which has O for 1 of its points.

Given: A line m and a hyperplane OABC that are not $\perp$.

To Prove: The projection of a line m upon the hyperplane OABC is a line c or a part of c.

Proof: Let a be the $\perp$ projecting ~~some~~ point P of m upon the hyperplane OABC in a point E. Any other $\perp$ b projecting a point Q of m upon the hyperplane OABC in a point F, lies in a plane with a (Art. ~~12~~ 13, Th. 2), and this plane containing 2 points of m, is the plane of ma determined by m and a, and the projection upon the hyperplane OABC is the same as its projection upon a line c in which the plane of ma intersects the hyperplane OABC. The line c is then the line of EF. For this reason, the theorem is therefore a theorem in the plane-geometry of the plane ma. Therefore the projection of a line m upon the hyperplane OABC is a line c or a part of c. (Q.E.D)

THEOREM 2.
~~Corollary~~. When a $\frac{1}{2}$-line drawn from a point O of a hyperplane does not lie in the hyperplane and is not perpendicular to it, the angle which it makes with the $\frac{1}{2}$-line drawn from O containing its projection, is less than the angle which it makes with any other $\frac{1}{2}$-line drawn in the hyperplane from O.

When a $\frac{1}{2}$-line drawn from a point O of a hyperplane does not lie in the hyperplane and is not perpendicular to it, the angle which it makes with its projection is called the ANGLE OF THE $\frac{1}{2}$-LINE AND HYPERPLANE. A $\frac{1}{2}$-line drawn from a point O of a hyperplane perpendicular to the hyperplane is said to make a RIGHT-ANGLE with the hyperplane.

II ABSOLUTELY-PERPENDICULAR PLANES

15. EXISTENCE OF ABSOLUTELY-PERPENDICULAR PLANES.

Notation: The symbol for absolutely-perpendicular is $\underline{\vee}$.

Theorem 1. A plane has more than 1 line perpendicular to it at a given point. (see Fig. 18.)

Given: A plane OAB and a point O of this plane.

To Prove: The plane OAB has more than 1 line $\perp$ to it at a point O.

Proof: The plane OAB is the intersection of different hyperplanes. Let OABC and OABP be 2 hyperplanes that intersect in the plane OAB. Now in a hyperplane, a plane has 1 and only 1 line $\perp$ to it at a given point. In the hyperplanes OABC and OABP, construct the lines OC and OP $\perp$ to the plane OAB at the point O, respectively (see Art. 1). We have, then, 2 lines OC and OP $\perp$ to the plane OAB at O. Therefore the plane OAB has more than 1 line $\perp$ to it at the point O. (Q.E.D)

Theorem 2. 2 lines perpendicular to a plane at a given point determine a 2nd plane, and the 2 planes are so related that every line of one through the point is perpendicular to every line of the other through the point. (Fig. 20.)

Given: A plane α and 2 lines p and q $\perp$ to α at a point O, with a plane α' determined by the lines p and q, and any line c of α passing through the point O, and any line r of α' passing through the point O.

To Prove: Any line c of α passing through the point O is $\perp$ to any line r of α' passing through the point O.

Proof: The lines p and q being $\perp$ to a plane α at a point O, are $\perp$ to all lines of α which pass through the point O (Follows from a repeated use of a theorem of the solid-geometry, where a line $\perp$ to a plane at a point O is $\perp$ to all lines of the plane which pass through the point O.). Now c being $\perp$ to p and q at O, is $\perp$ to every line through O in the plane α' which the 2 lines p and q determine (Same reference to a theorem of the solid-geometry as stated above.). Therefore c is $\perp$ to any line r in α' passing through the point O. But c is any line of α passing through O. Therefore any line c of α

passing through the point O is $\perp$ to any line r of α' passing through the point O.

Theorem 3. All the lines perpendicular to a plane at a given point lie in a single plane. (Fig. 21.)

Given: A plane α and a point O of α, and any line a $\perp$ to α at the point O.

To Prove: Any line a $\perp$ to α at a point O lies in a plane α'.

Proof: 2 lines p and q $\perp$ to α determine a plane α' in which every line through O is a line $\perp$ to α. Now a being any line $\perp$ to α at the point O. The hyperplane determined by a and α intersects α' in a line b (Art. 4, Th. 1), also $\perp$ to α at O. But in a hyperplane containing α only 1 line can be $\perp$ to α at O. For this reason, a coincides with b and lies in α'. Therefore any line a $\perp$ to α at a point O lies in a plane α'. (Q.E.D)

2 planes having a point in common are ABSOLUTELY-PERPENDICULAR when every line of one through that point is perpendicular to every line of the other through that point.

These planes have only the point in common, and do not intersect in a line. We can never see both planes in a single hyperplane like the space in which we live. The most that we could see would be 1 plane and a single line of the other. (see Fig. 21.)
The black-line-renderings lie in the hyperplane of pα, we see the plane α, and only the line p of α' <u>in</u> the hyperplane of pα.

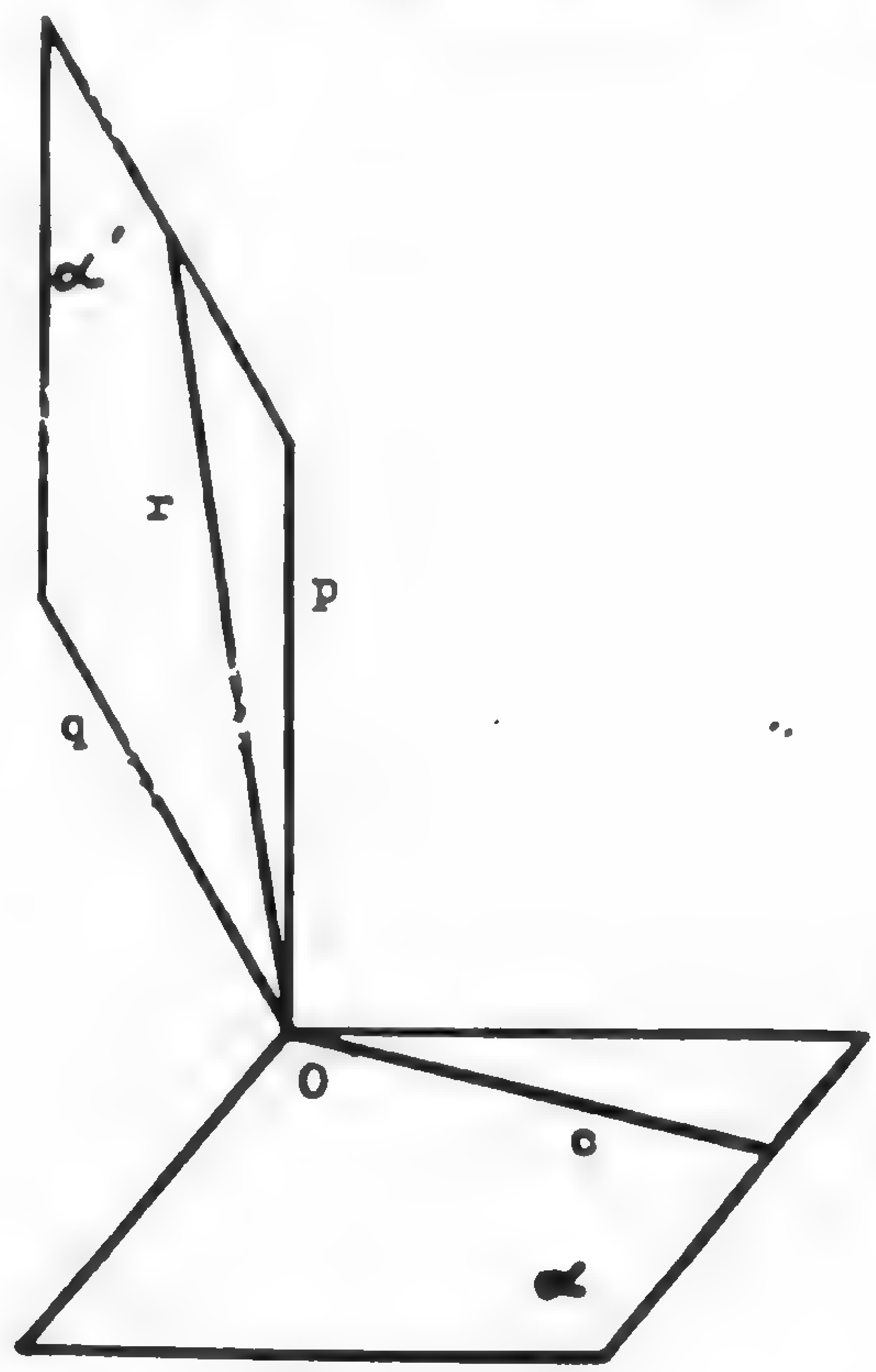


Fig. 20.

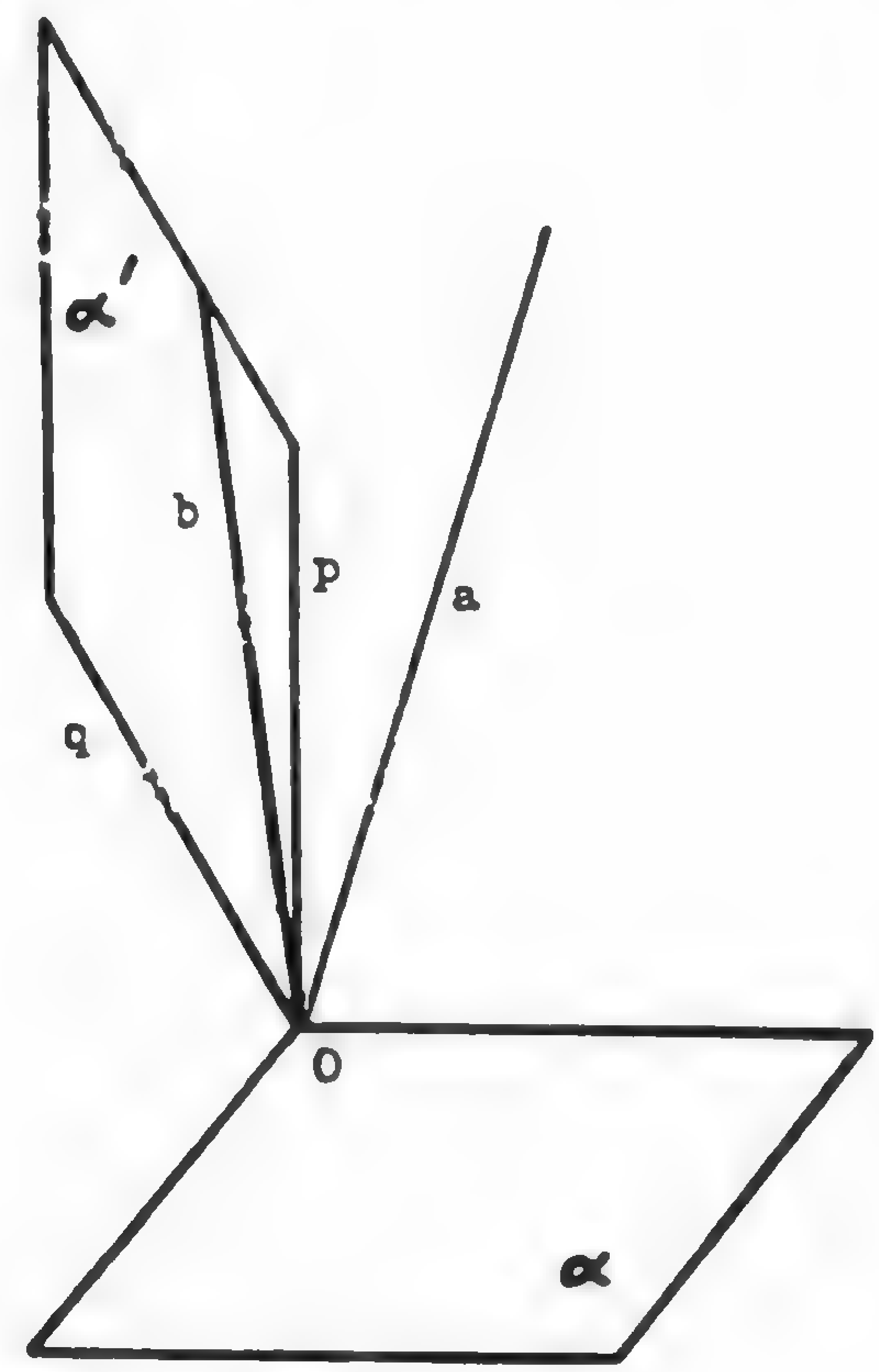


Fig. 21.

16. 1 PLANE THROUGH ANY POINT ABSOLUTELY-PERPENDICULAR TO A GIVEN PLANE.

Theorem 1. At any point of a plane there is 1 and only 1 plane absolutely-perpendicula to a given plane.

Use the theorems of the preceding article for the proof of this theorem.

A plane and a point outside of a plane lie in 1 and only 1 hyperplane. Through any such point, therefore, by solid-geometry, passes 1 and only 1 line perpendicular to the plane; and the PROJECTION OF THE POINT UPON THE PLANE, as in solid-geometry, is the foot of this perpendicular. A point that lies in a given plane is its own projection upon the plane.

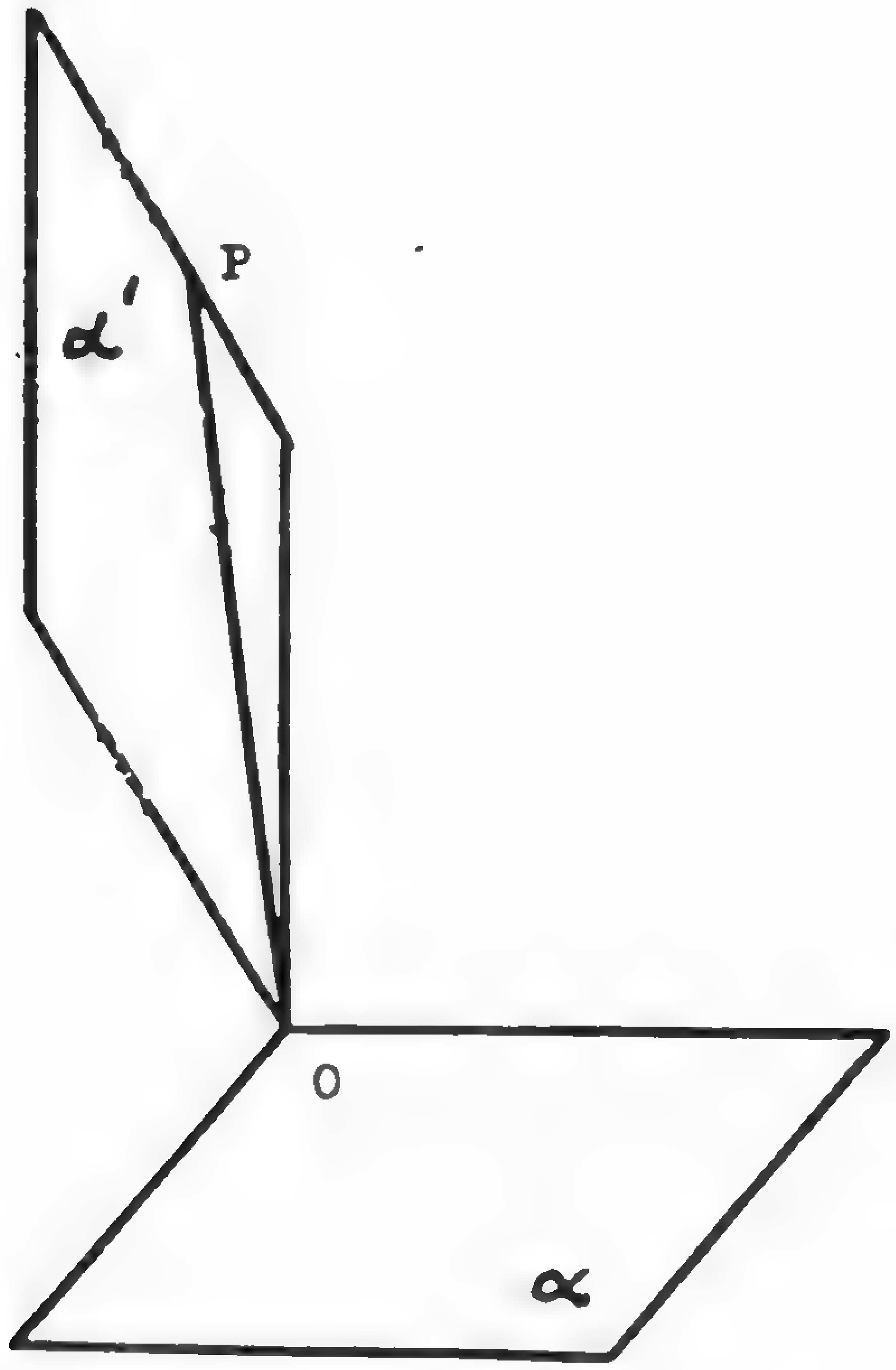


Fig. 22.

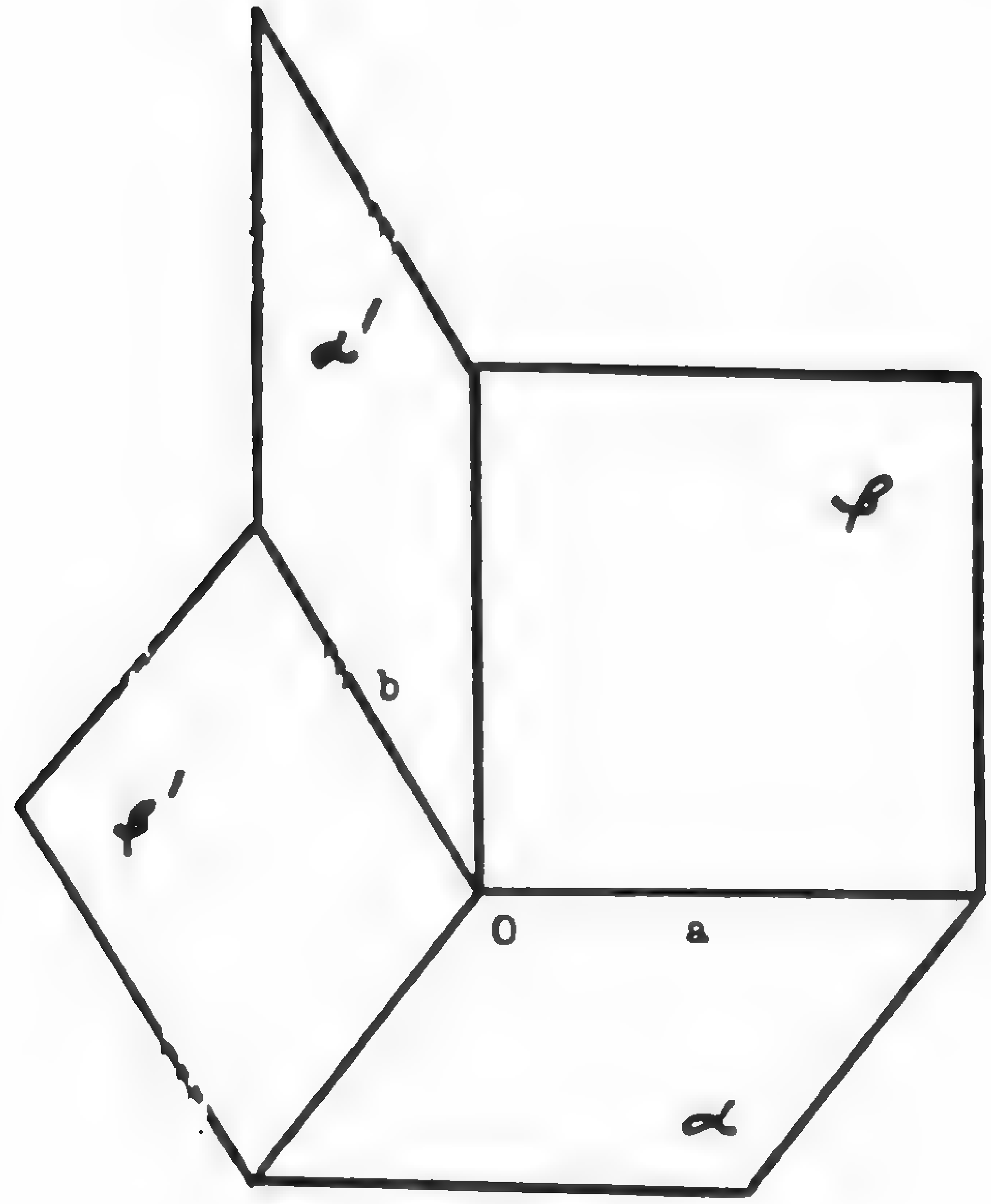


Fig. 23.

Theorem 2. Through any point outside of a plane passes 1 and only 1 plane absolutely-perpendicular to the given plane. (Fig. 22.)

Given: A point P outside of a plane α.

To Prove: A plane α' passing through the point P is the only plane $\underline{Y}$ to α.

Proof: Let the projection of the point P on the plane α be O. The $\underline{Y}$ plane α' at O will then contain the $\perp$ line OP, and therefore P. Moreover, any $\underline{Y}$ plane containing P must contain a $\perp$ line through P, and there is only 1 such line. Therefore a plane α' passing through the point P is the only plane $\underline{Y}$ to α. (Q.E.D)

17. PLANES ABSOLUTELY-PERPENDICULAR TO PLANES WHICH INTERSECT IN A LINE.

Theorem 1. If 2 planes intersect in a line and so lie in a hyperplane, their absolutely-perpendicular planes at any point of their intersection intersect in a line and lie in a hyperplane. (Fig. 23.)

Given: 2 planes α and β intersecting in a line a, and the $\underline{Y}$ planes α' and β' of α and β respectively, at a point O of a of their intersection.

To Prove: The planes α' and β' $\underline{Y}$ to α and β respectively, at a point O of a of their intersection, intersect in a line b and lie in a hyperplane.

Proof: α' and β' are both $\perp$ to the line a at O, and lie in a hyperplane $\perp$ to a at O (Art. 12, Th. 3). Therefore α' and β' intersect in a line b (see Th.1 of RSG-III). Therefore the planes α' and β' $\underline{Y}$ to α and β respectively, at a point O of a of their intersection, intersect in a line b and lie in a hyperplane. (Q.E.D)

Theorem 2. If 3 planes have a line in common, their absolutely-perpendicular planes at any point of this line lie in a hyperplane; and if 3 planes lie in a hyperplane and have a point in common, their absolutely-perpendicular planes at this point have a line in common.

The proof of this theorem follows from Theorem 3 of Art. 12, the line being perpendicular to the hyperplane.

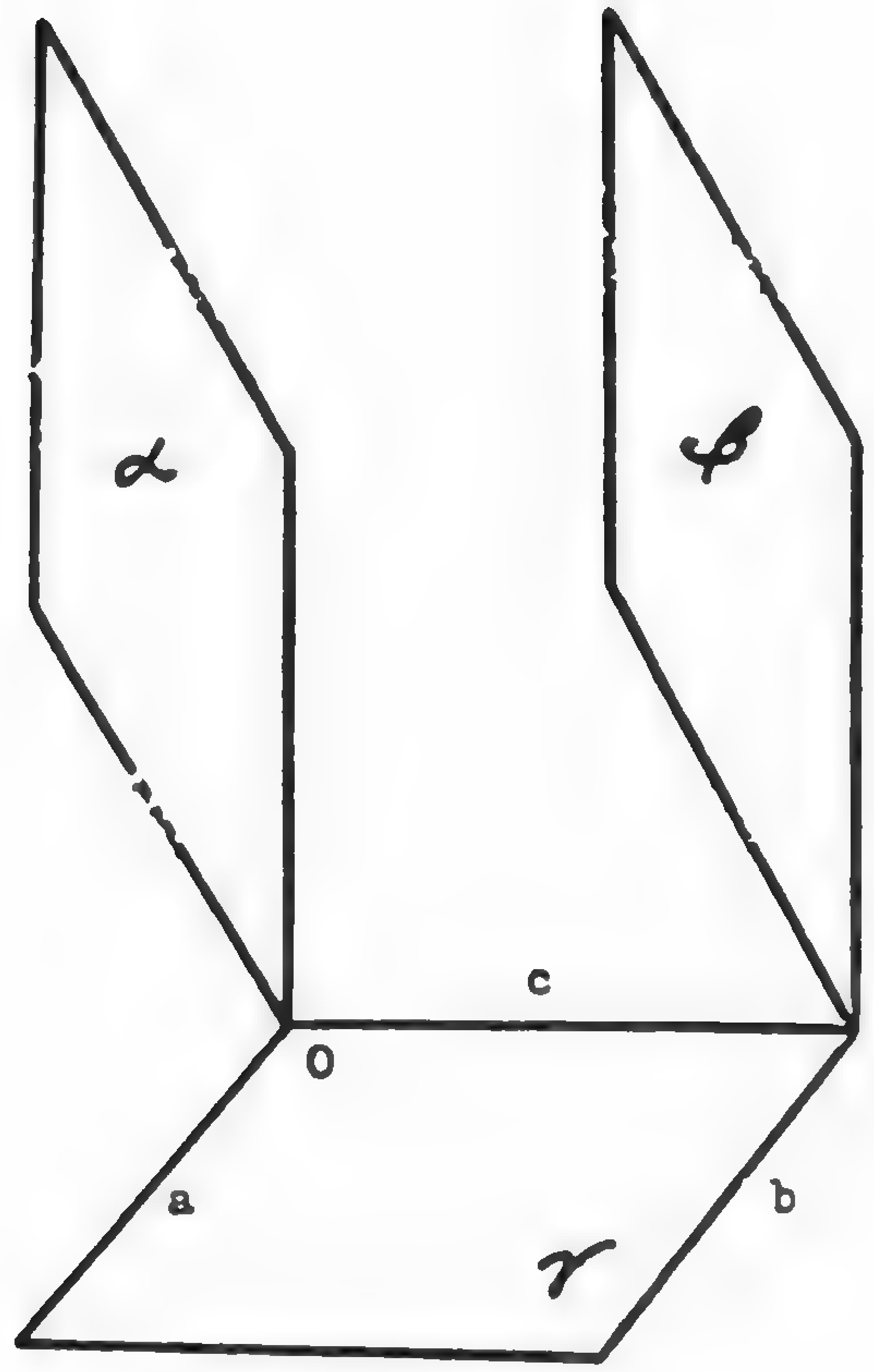


Fig. 24.

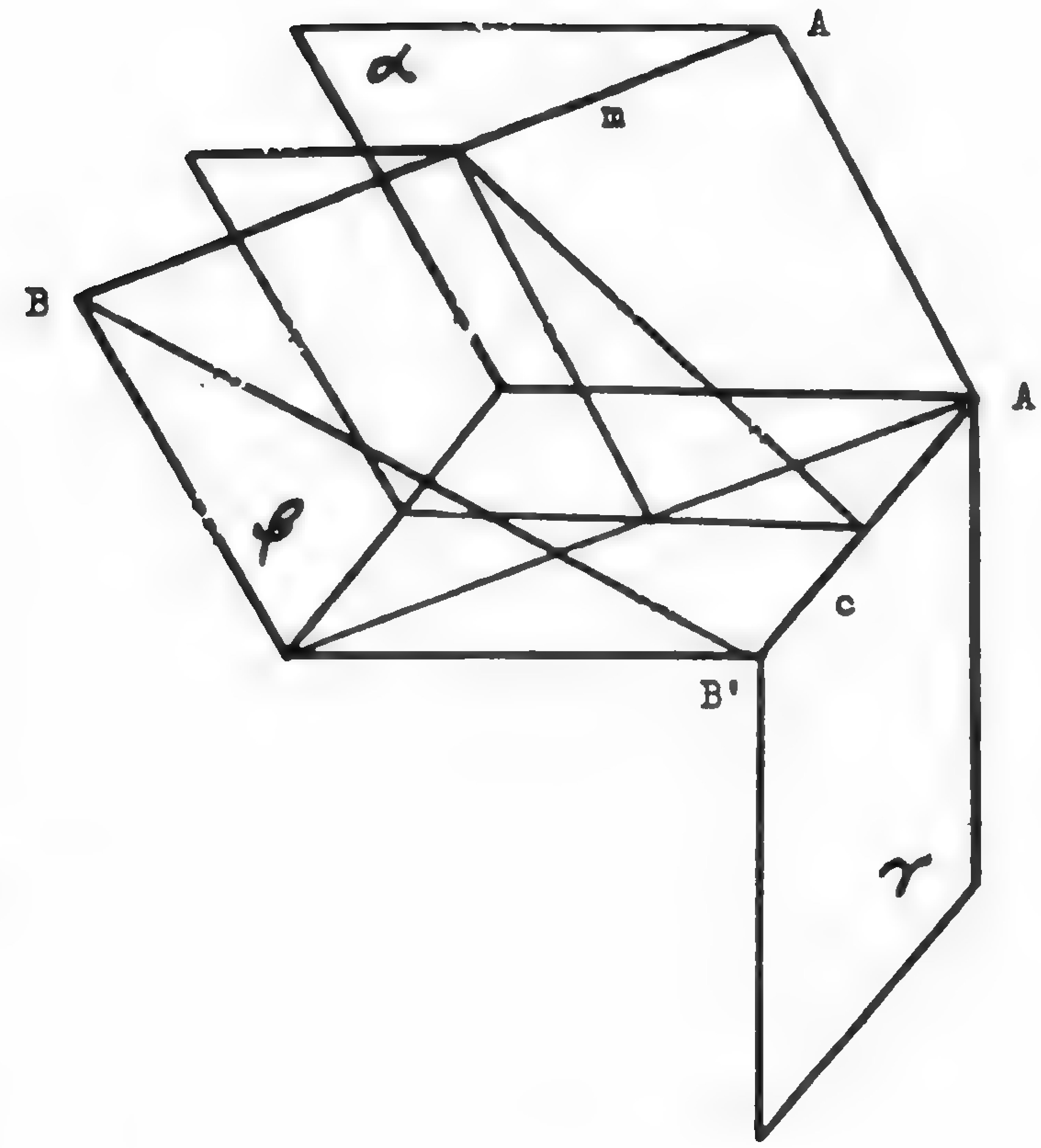


Fig. 25.

Corollary. If 3 planes have a line in common and lie in a hyperplane, their absolutely-perpendicular planes at any point of this line have a line in common and lie in a hyperplane.

18. PLANES ABSOLUTELY-PERPENDICULAR TO A PLANE AT DIFFERENT POINTS.

Theorem. 2 planes absolutely-perpendicular to a 3rd lie in a hyperplane. (Fig. 24.)

Given: 2 planes α and β $\underline{\perp}$ to a 3rd plane γ at points O and O' respectively.

To Prove: The planes α and β lie in the hyperplane determined by α and the line c of OO'.

Proof: Let a and b be the lines $\perp$ at O and O' to the hyperplane determined by α and c. a and b lie in a plane (Art. 13, Th. 2), and this plane is γ, for a being $\perp$ to α, must lie in γ, and only 1 plane can contain a and the point O'. b is then $\perp$ to β; so that β lies with α in the hyperplane to which b is $\perp$ (Art. 12, Th. 3), the hyperplane determined by α and c. Therefore the planes α and β lie in the hyperplane determined by α and the line c of OO'. (Q.E.D)

Corollary. All the planes absolutely-perpendicular to a plane at the points of any line of it lie in a hyperplane.

The most that we could see <u>in</u> any 1 hyperplane would be the plane γ and a line in each of the other 2 planes, or the 2 planes α and β and a single line c of γ.

19. PROJECTION OF A LINE UPON A PLANE. As in other cases of projection, the projection of any figure upon a plane consists of the projections of its points. (Art. 14).

Theorem 1. The perpendiculars projecting the points of a line upon a plane do not lie in a single plane, unless the line itself lies in a hyperplane with the plane upon which it is projected.

Proof: If any 2 of the $\perp$s were in a plane, that plane, having 2 points in common with the given plane would intersect the latter in a line and lie with it in a hyperplane (Art. 1, Th. 2 (4)). Therefore, the given line would lie in a hyperplane with the plane

upon which it is projected. (See Fig. 25, and use the plane of mA' lying in the hyperplane ABA'B'; complete the details of the graphic, then the results are quite obvious.

When 2 planes are absolutely-perpendicular at a point O, all the points of one project upon the other in a single point O. We may regard projection upon a plane as made by planes absolutely-perpendicular to it, just as in solid-geometry we may regard projection upon a line as made by planes perpendicular to the line. We shall sometimes speak of PROJECTING-PLANES and think of a point as projected upon a plane in this way.

Theorem 2. The projection of a line upon a plane is a line or a part of a line, unless the given line lies in a plane absolutely-perpendicular to the given plane. (Fig. 25.)

Given: A line m and a plane γ.

To Prove: The projection of the line m upon the plane γ is a line c or a part of c.

Proof: Let α be the plane $\underline{Y}$ to γ which projects some point A of m upon γ. Any other plane β projecting a point B of m upon γ lies in a hyperplane with α (Art. 18, Th.) But this hyperplane containing 2 points of m, is the hyperplane determined by m and α; and the projection of m upon γ is the same as its projection upon the line c in which this hyperplane intersects γ. Therefore the projection of the line m upon the plane γ is a line c or a part of c. (Q.E.D)

The projecting-lines form a RULED-SURFACE which contains the given line and its projection and lies in the hyperplane of these 2 lines. (See Fig. 25, the 2 lines m and c determine the ruled-surface in the hyperplane of mc.)

III SIMPLY-PERPENDICULAR PLANES

20. PLANES INTERSECTING IN A LINE 2 ABSOLUTELY-PERPENDICULAR PLANES. 2 planes are PERPENDICULAR, or SIMPLY-PERPENDICULAR, when they lie in 1 hyperplane and in this hyperplane form right-dihedral-angles. Each contains a line in the other and lines perpendicular to the other.

In one sense, 2 planes are perpendicular whenever one of them contains a line perpendicular to the other, such planes have also been called $\frac{1}{2}$-perpendicular. 2 planes in a hyperplane forming a right-dihedral-angle can be decribed as PERPENDICULAR IN A HYPERPLANE. We shall use the sense of $\frac{1}{2}$-perpendicular planes in a hyperplane, in the ordinary-sense as used in our solid-geometry, i.e. we shall call the $\frac{1}{2}$-perpendicular planes in a hyperplane forming a right-dihedral-angle, simply, 'perpendicular' planes in a hyperplane.

Theorem 1. A plane perpendicular to 1 of 2 absolutely-perpendicular planes, and passing through the point where they meet, is perpendicular to the other. (Fig. 26.)

Given: 2 $\underline{Y}$ planes α and α' that meet in a point O, and β a plane passing through O and $\perp$ to α.

To Prove: β is $\perp$ to α'.

Proof: 2 planes which are $\perp$ lie in a hyperplane, and by a theorem of solid-geometry, a line in one $\perp$ to their intersection is $\perp$ to the other. There is, then, a line d $\perp$ to α at O, that is, a line d common to β and the $\underline{Y}$ plane α', so that β intersects α' in a line d and lies with α' in a hyperplane. Now the line c in which β intersects α, like all the other lines of α through O, is $\perp$ to α'; so in the hyperplane of β and α' we have a line c lying in β $\perp$ to α'. Therefore β is $\perp$ to α'. (Q.E.D)

Theorem 2. A plane intersecting in a line each of 2 absolutely-perpendicular planes, is perpendicular to both. (see Fig. 26.)

Given: 2 $\underline{Y}$ planes α and α' that meet in a point O, and β a plane intersecting α and α' in lines c and d respectively.

To Prove: β is $\perp$ to α and α'.

Proof: The lines c and d, and β pass through O; otherwise the hyperplane determined by β and O would contain both α and α', which is impossible. Now the line d in which β

intersects α' is that line which is $\perp$ to α at O in the hyperplane determined by α and β. In the hyperplane of $\alpha\beta$, then, we have a line d lying in β and $\perp$ to α. In the same way, or by Th. 1, we prove that β is $\perp$ to α'. Therefore β is $\perp$ to α and α'. (Q.E.D)

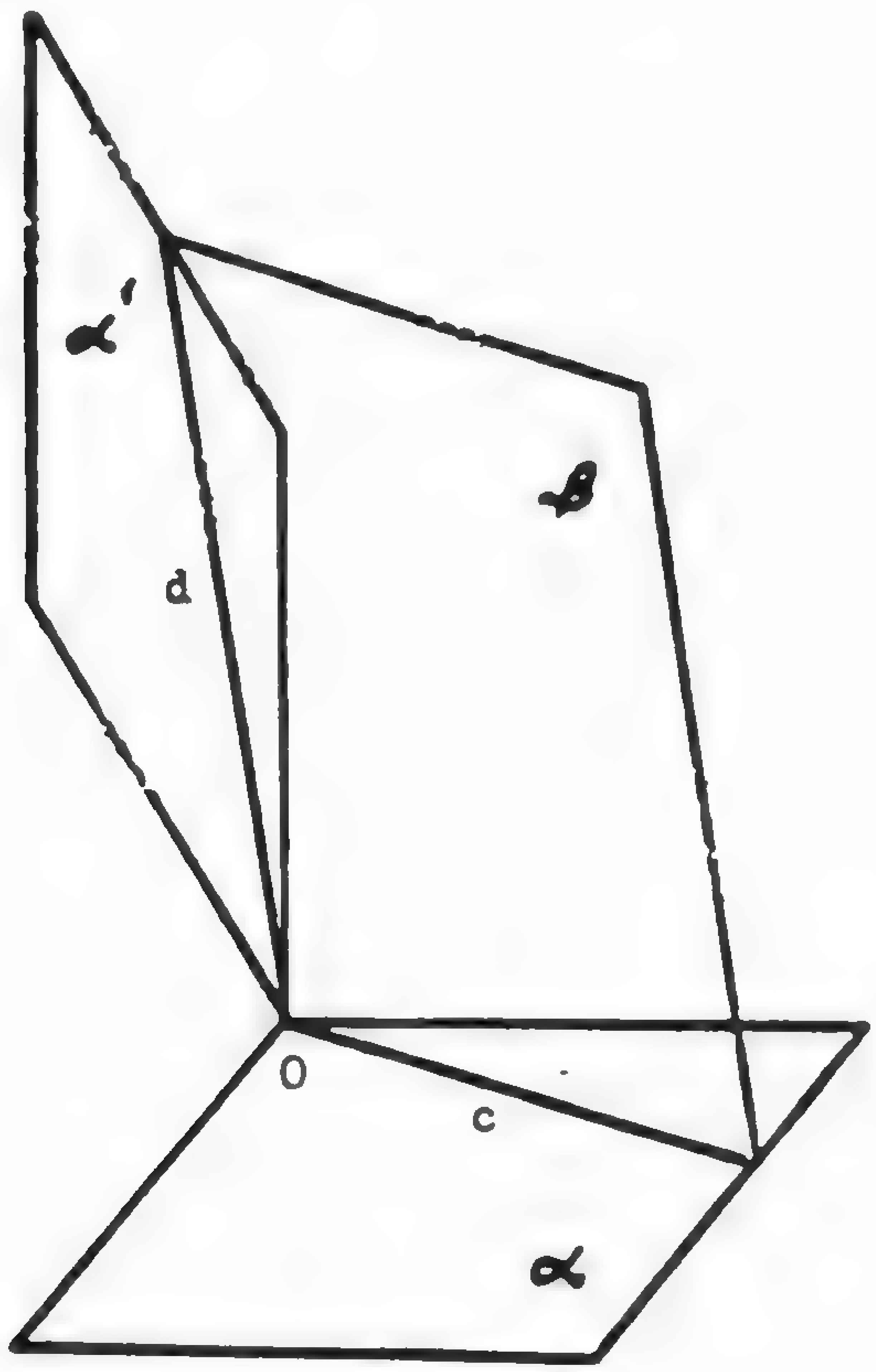


Fig. 26.

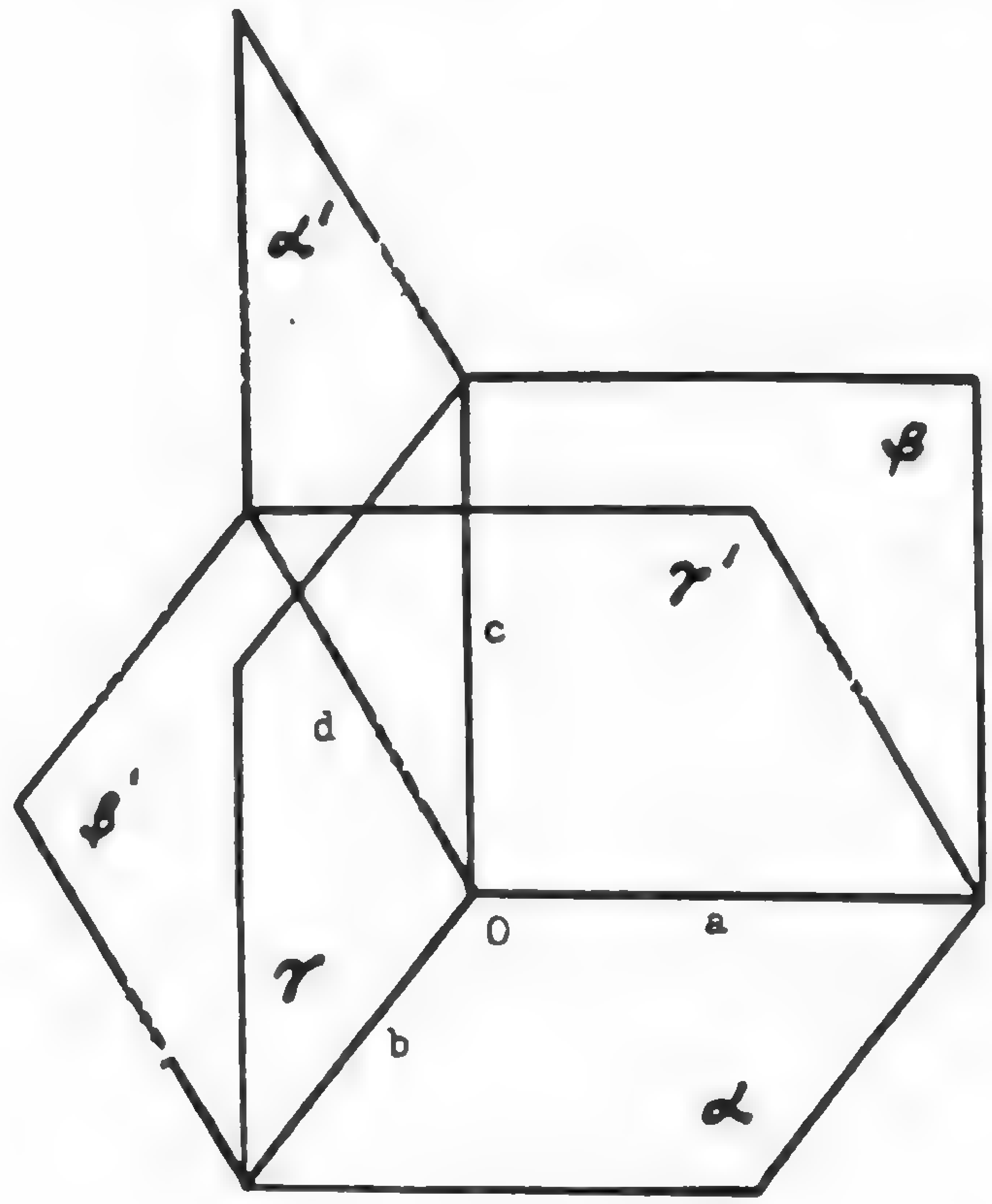


Fig. 27.

Theorem 3. If 2 planes are perpendicular, their absolutely-perpendicular planes at any point of their intersection are perpendicular. (Fig. 23.)

Given: 2 $\perp$ planes α and β that intersect in a line a, and α' and β' their ⊻ planes at a point O of their line of intersection a.

To Prove: α' and β' are $\perp$ at a point O of the line of intersection a of α and β.

Proof: α, being $\perp$ to β, is $\perp$ to β'; and then β', being $\perp$ to α, is $\perp$ to α' (Th. 1 of this article). Therefore α' and β' are $\perp$ at a point O of the line of intersection a of α and β. (Q.E.D)

21. 2 PAIRS OF ABSOLUTELY-PERPENDICULAR PLANES AT A POINT—Possible-Positions. Let 2 pairs of absolutely-perpendicular planes have their intersection point O in common. Then 1 of 3 possibilities may occur as follows:

(1) they may have only the point O in common;
(2) each plane of 1 pair may intersect in a line 1 plane of the other pair;
(3) each plane of 1 pair may be perpendicular to both planes of the other pair.

In (3) the 4 lines of intersection are mutually-perpendicular, and, taken 2 at-a-time, determine also a 3rd pair of absolutely-perpendicular planes. The planes of each of the 3 pairs are then perpendicular to all the planes of the other 2 pairs.

Actually, we have 4 mutually-perpendicular lines, such that, any 2 of them will determine a plane, and any 3 will determine a hyperplane. Each line is perpendicular to the hyperplane determined by the other 3. In each hyperplane are 3 of the 6 planes, 3 mutually-perpendicular planes forming a trirectangular-trihedral-angle.

We shall call this figure a RECTANGULAR-SYSTEM (see Art. 40).

Note: In general, we shall use the word 'pair' in speaking of 2 planes only when we have in mind 2 planes absolutely-perpendicular to each other. Thus we shall speak in-this-way of a pair of planes perpendicular to a given plane, and of a pair of common-

perpendicular planes when we have 2 given planes.

22. COMMON-PERPENDICULAR PLANES OF 2 PLANES INTERSECTING IN A LINE.

Theorem. 2 planes which intersect in a line have at any point O of that line 1 and only 1 pair of common-perpendicular planes. (Fig. 27.)

Given: 2 planes α and β which intersect in a line a.

To Prove: α and β have 1 and only 1 pair of common $\perp$ planes at any point O of their line of intersection a.

Proof: Let α' and β' be the planes $\underline{Y}$ to α and β at O. A plane which is $\perp$ to α and β at O is a plane which intersects these 4 planes in lines. The planes α and β intersect in a line a and lie in a hyperplane, and the planes α' and β' intersect in a line d and lie in a hyperplane (Art. 17, Th. 1). d is $\perp$ to the hyperplane of α and β, and a is $\perp$ to the hyperplane of α' and β' (Art. 12, Th. 4).

In a hyperplane 2 intersecting planes have a common $\perp$ plane at any point of their intersection, $\perp$ to the intersection. In the hyperplane of α and β, let γ be the common $\perp$ plane to α and β at a point O of their line of intersection a, $\perp$ to a. γ cannot contain the intersection of α' and β', for d, the intersection of α' and β', is $\perp$ to the hyperplane of α and β. γ must therefore intersect α' and β' in separate lines b and c respectively, and lie in their hyperplane. That is to say, the common $\perp$ plane with which we are familiar in the case of 2 intersecting planes, $\perp$ to their intersection at a point O, is the plane of intersection of their hyperplane with the hyperplane of their $\underline{Y}$ planes at O.

A 2nd common $\perp$ plane γ' is the plane $\underline{Y}$ to the plane γ (Art. 20, Th. 1). γ' contains the lines a and d which are the lines of intersection α and β and of α' and β' respectively, and may be regarded as determined by them.

We now prove that γ and γ' are the only planes $\perp$ to α and β at O. Any plane intersecting α and β in lines must pass through their line of intersection or lie in their hyperplane, and any plane intersecting α' and β' in lines must pass through their line of intersection or lie entirely in their hyperplane (Art. 1, Th. 1). But, as we have seen, a plane lying entirely in the hyperplane of 2 of these planes cannot pass through the line of intersection of the other 2. Any common $\perp$ plane, therefore, must lie in both hyperplanes or pass through both lines of intersection: that is, it must be 1 of the 2 planes already found, and this plane is γ. γ and γ' are therefore the only planes which can be $\perp$ to α and β at O. Therefore α and β have 1 and only 1 pair of common $\perp$ planes at any point O of their line of intersection a. (Q.E.D)

Suppose the given planes α and β are $\perp$, then the 4 planes α, β, α', and β' have all together the 4 lines of intersection, a, b, c, and d. We have 4 hyperplanes that meet in a point O, and which contain 2 of the 4 planes. These 4 planes can be associated in 2-ways so that 2 planes shall intersect in a line and the other 2 planes be their $\underline{Y}$ planes. But the plane determined by the 2 lines of intersection in one case is the plane of intersection of the 2 hyperplanes in the other case. Therefore we get only 2 planes $\perp$ to the 2 given planes α and β.

Now, if α and β are $\perp$, with $\alpha \perp$ to β' and $\beta \perp$ to α', then the plane γ' determined by the line of intersection a of α and β and the line of intersection d of α' and β' is the plane of intersection γ' of the hyperplane determined by α and β' and the hyperplane determined by α' and β; and the plane γ determined by the line of intersection b of α and β' and the line of intersection c of α' and β is the plane of intersection γ of the hyperplane determined by α and β and the hyperplane determined by α' and β'. Thus we have only 2 planes γ and γ' intersecting in lines the 4 planes, and so $\perp$ to the planes α and β.

The 4 planes, and the 2 common $\perp$ planes in this case are the 6 planes of a rectangular-system (Art. 21).

Corollary. If each of 2 planes having a common point O intersect in a line the plane absolutely-perpendicular to the other at O, then these planes have 1 and only 1 pair of common-perpendicular planes.

23. PERPENDICULAR PLANES AND HYPERPLANES. THE PLANES PERPENDICULAR OR ABSOLUTELY-PERPENDICULAR TO PLANES LYING IN THE HYPERPLANE. A plane intersecting a hyperplane is PERPENDICULAR TO THE HYPERPLANE at a point of their intersection, if the plane absolutely-perpendicular to the given plane at this point lies in the hyperplane. The hyperplane is also said to be PERPENDICULAR TO THE PLANE.

Notation: We shall designate a hyperplane by the Greek capital-letter X, and we shall call X, 'Chi'; the symbol X designating a hyperplane, will be put near 1 of the corners of a 'parallelopiped' representing the hyperplane. From 1 to 3 hyperplanes, we shall use the last 3 capital-letters of the Greek-alphabet, i.e. X, Ψ, and Ω, which are called respectively, 'Chi', 'Psi', and 'Omega'. For more than 3 hyperplanes, we shall use different Greek capital-letters as the need arises.

Theorem 1. If a plane is perpendicular to a hyperplane at 1 point of their line of intersection, it is perpendicular to the hyperplane. (Fig. 28.)

Given: A plane α and a hyperplane X intersecting in a line c, with $\alpha \perp$ to X at a point O of their line of intersection c.

To Prove: α is $\perp$ to the hyperplane X at all points of their line of intersection c.

Proof: Let β be the plane $\underline{Y}$ to α at O, then the line of intersection c and the plane $\underline{Y}$ to α at O determine the hyperplane X (Art. 1, Th. 2(1)), which, therefore, contains the planes $\underline{Y}$ to the plane α at all points of their line of intersection c (Art. 18, Th. and Cor.). Therefore α is $\perp$ to the hyperplane X at all points of their line of intersection c. (Q.E.D)

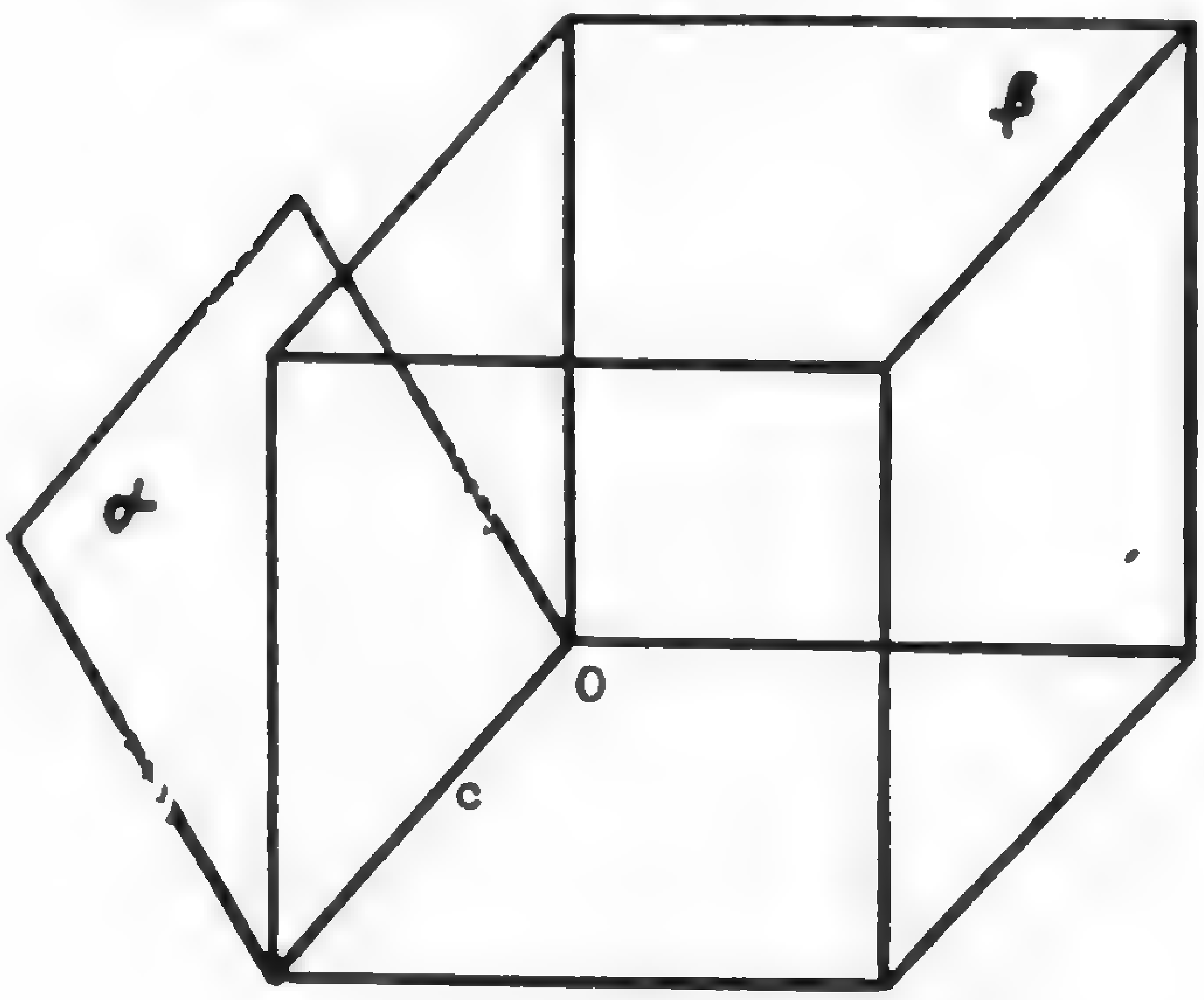


Fig. 28.

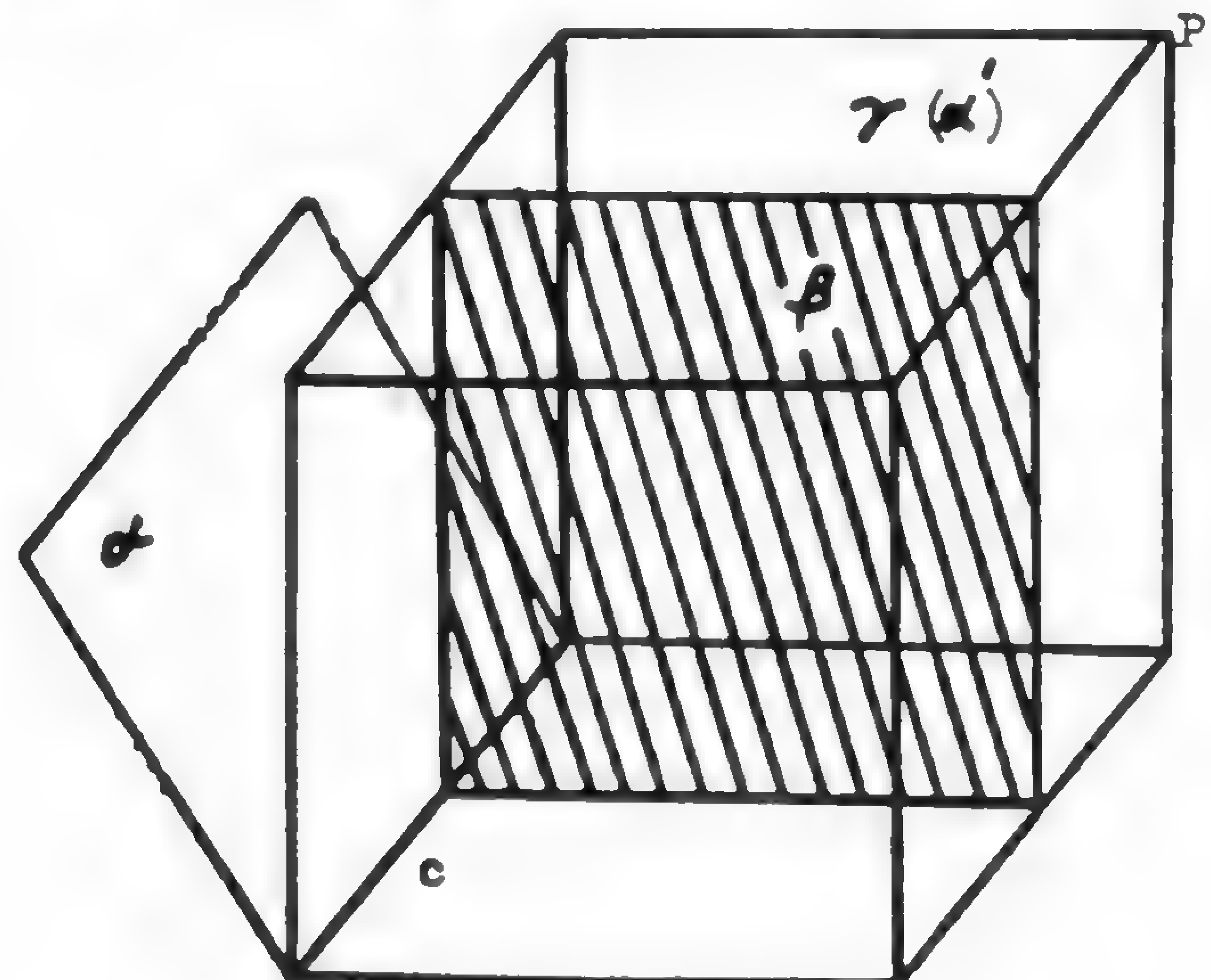


Fig. 29.

Theorem 2. If a plane α is perpendicular to a hyperplane along a line c, any plane in the hyperplane perpendicular to c is absolutely-perpendicular to α, and any plane absolutely-perpendicular to α through a point of the hyperplane lies entirely in the hyperplane. (Fig. 29.)

Case 1. Given: A plane $\alpha \perp$ to a hyperplane X along a line c, and any plane β in the hyperplane X $\perp$ to c.

To Prove: β is $\underline{Y}$ to α.

Proof: The planes $\underline{Y}$ to α at the points of c lie in the hyperplane X, and in X there are planes $\perp$ to the line c. Now in the hyperplane only 1 plane can be $\perp$ to a given line at a given point. Any plane β in X, therefore, $\perp$ to the line c, must be 1 of the planes $\underline{Y}$ to α at the points of c. Therefore β is $\underline{Y}$ to α. (Q.E.D)

Case 2. Given: A plane $\alpha \perp$ to a hyperplane X along a line c, and any plane α' $\underline{Y}$ to through a point P of the hyperplane X.

To Prove: α' lies entirely in X.

Proof: In the hyperplane X, we can draw a plane γ in X through the same point P $\perp$ to c. γ is then $\underline{V}$ to α, and α' must coincide with γ and lie entirely in the hyperplane X; since we cannot have 2 planes through a point $\underline{V}$ to a given plane (Art. 16, Ths.). Therefore α' lies entirely in X. (Q.E.D)

Theorem 3. If a plane α is perpendicular to a hyperplane along a line c, any plane in the hyperplane passing through c is perpendicular to α, and any plane perpendicular to α passing through c, or through any line which lies in the hyperplane and is not itself perpendicular to c, lies entirely in the hyperplane. (Fig. 30.)

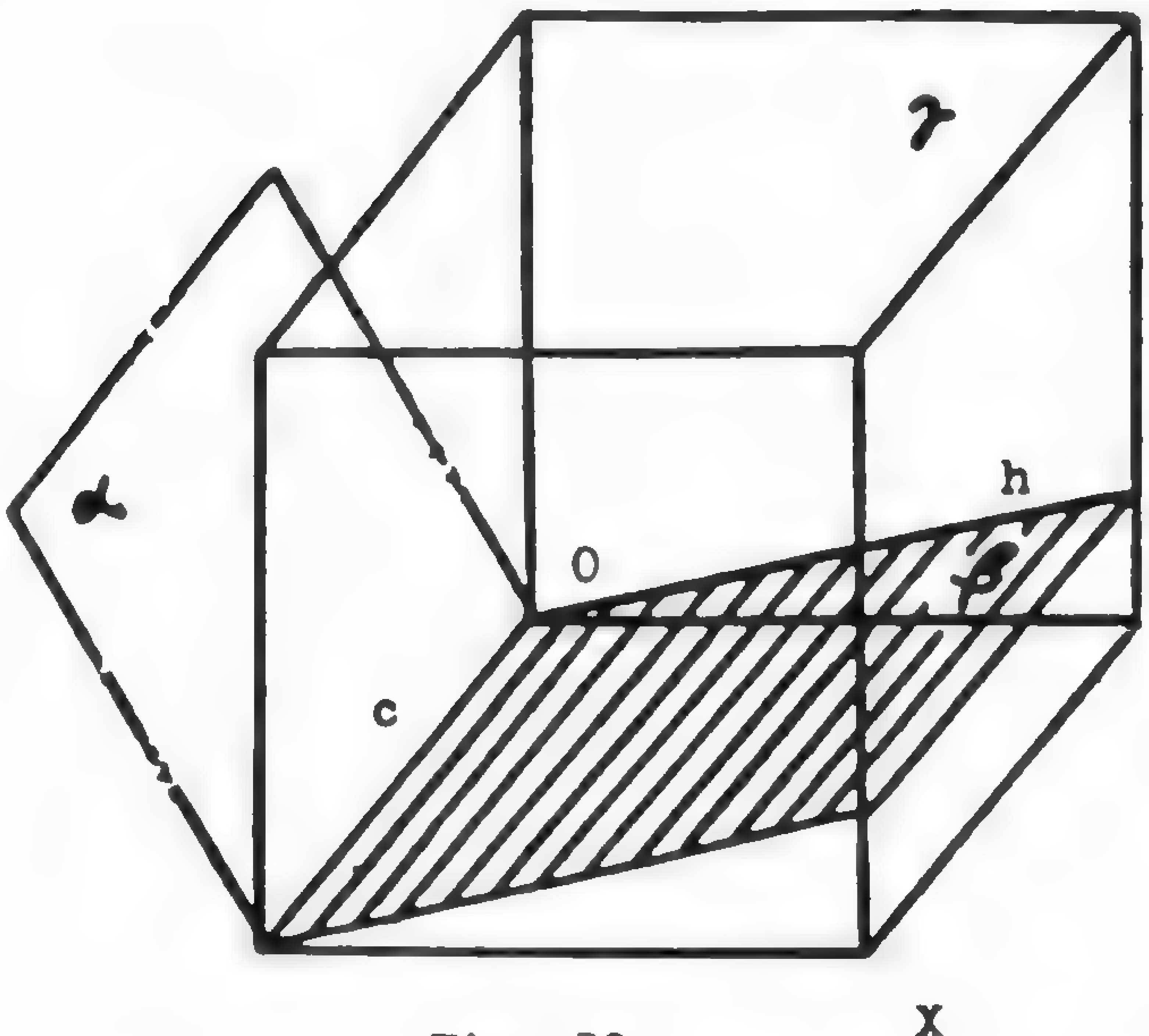


Fig. 30.

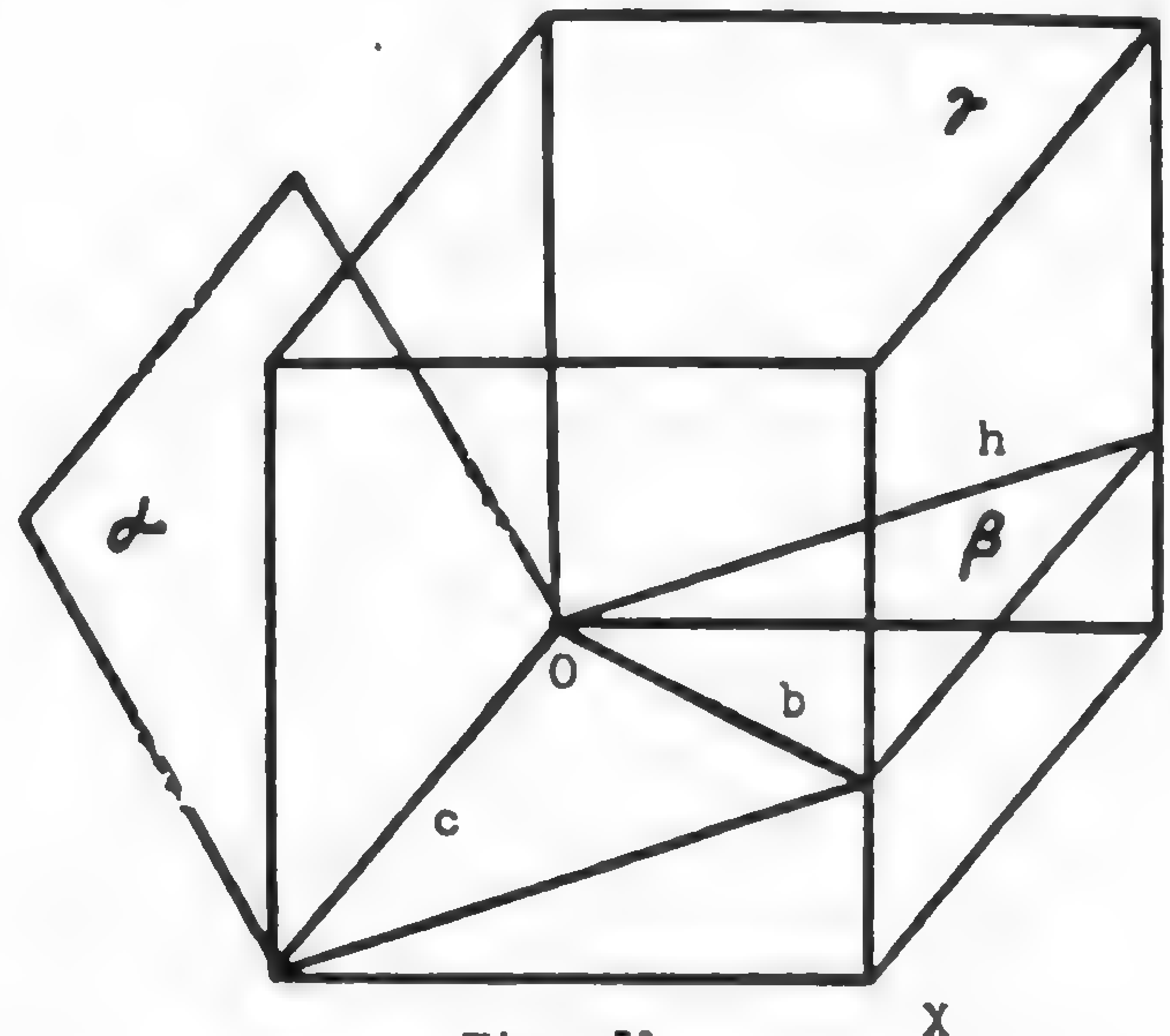


Fig. 31.

Case 1. Given: A plane α $\perp$ to a hyperplane X along a line c, and any plane β in X passing through c.

To Prove: β is $\perp$ to α.

Proof: Let γ be the plane $\underline{V}$ to α at a point O of c. Now β is given as passing through c, and therefore intersects α in the line c; and, lying in X, it intersects γ in a line h which is $\perp$ to α at O. It is therefore $\perp$ to α. Therefore β is $\perp$ to α. (Q.E.D)

Case 2a. Given: A plane α $\perp$ to a hyperplane X along a line c, and any plane β $\perp$ to α passing through c.

To Prove: β lies entirely in X.

Proof: Let γ be the plane $\underline{V}$ at a point O of c. Now β is given as $\perp$ to α and passing through c. β intersects γ in a line h (Th. 2). It must therefore contain 2 lines c and h of the hyperplane X and lie entirely in it (Art. 1, Th. 1). Therefore β lies entirely in X. (Q.E.D)

Case 2b. Given: A plane α $\perp$ to a hyperplane X along a line c, and any plane β passing through a line b which lies in X and is not itself $\perp$ to c. (Fig. 31.)

To Prove: β lies entirely in X.

Proof: Let γ be the plane $\underline{V}$ at a point O of c. Now β is given as passing through a line b which lies in X and is not itself $\perp$ to c. β intersects γ in a line h (Th. 2). It must therefore contain 2 lines b and h of the hyperplane X and lie entirely in it (Art. 1, Th. 1). Therefore β lies entirely in X. (Q.E.D)

Theorem 4. If 2 hyperplanes are perpendicular to a plane at a point O, they intersect in the absolutely-perpendicular plane at O.

For the absolutely-perpendicular plane lies in both hyperplanes, by hypothesis.

24. LINES LYING IN THE PLANE AND PERPENDICULAR TO THE HYPERPLANE, OR IN THE HYPERPLANE AND PERPENDICULAR TO THE PLANE.

Theorem 1. If a plane is perpendicular to a hyperplane, any line in the plane perpendicular to their intersection is perpendicular to the hyperplane, and any line perpendicular to the hyperplane through a point of the plane lies entirely in the plane. (Fig. 32.)

Case 1. Given: A plane $\alpha \perp$ to a hyperplane X, α intersecting X in a line c, and any line b in $\alpha \perp$ to c.

To Prove: b is $\perp$ to X.

Proof: Let Q be the point at which the line b is $\perp$ to c. Let γ be the plane $\perp$ to c at Q. b, lying in α, is not only $\perp$ to c, but also to the $\perp$ plane γ at the point Q where it meets c. The line b is therefore $\perp$ to the hyperplane X (Art. 12, Th. 4). Therefore b is $\perp$ to X. (Q.E.D)

Case 2. Given: A plane $\alpha \perp$ to a hyperplane X, α intersecting X in a line c, and any line b $\perp$ to X through a point P of α. (Fig. 33.)

To Prove: b lies entirely in α.

Proof: b is given as passing through a point P of α and $\perp$ to the hyperplane X, we can draw a line b' in α through the same point P $\perp$ to the intersection c, and the 2 lines b and b' must coincide, since they are both $\perp$ to X (Art. 13, Ths. 1 and 3). Therefore b lies entirely in α. (Q.E.D)

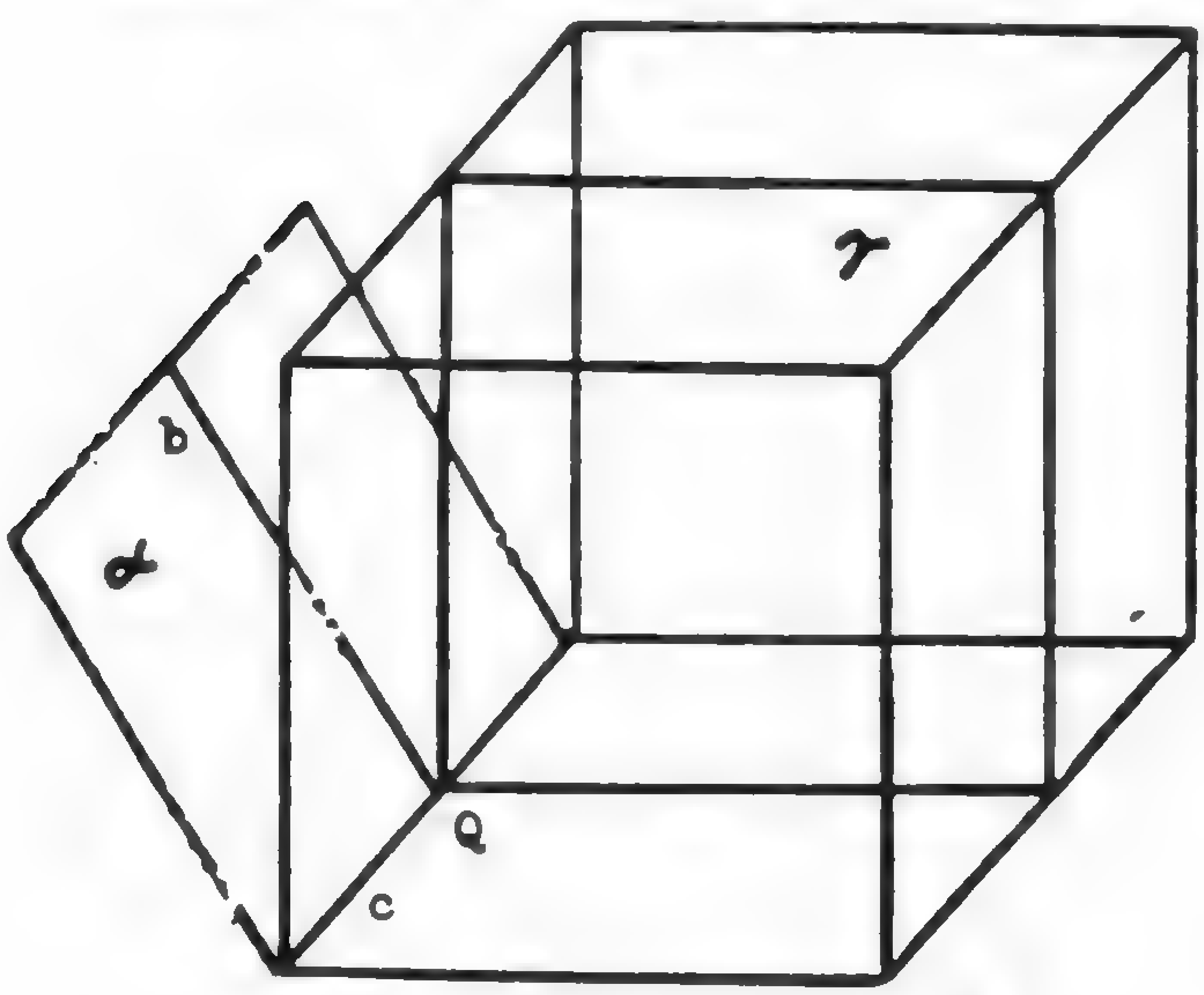


Fig. 32.

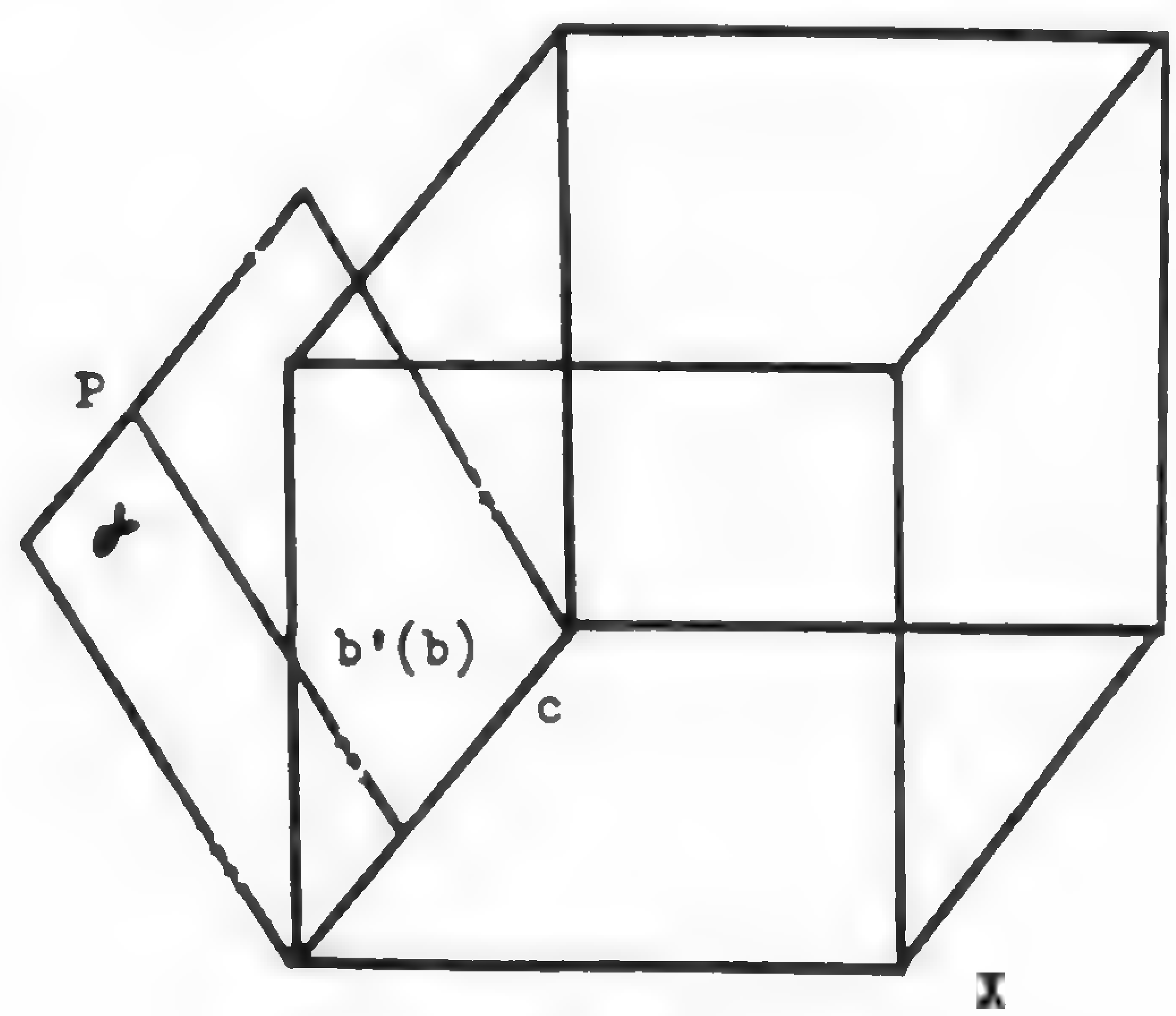


Fig. 33.

Corollary. If 2 planes are perpendicular to a hyperplane at a point O, they intersect in a line which is perpendicular to the hyperplane at O.

The corollary follows as an immediate consequence of the theorem, that is, the line perpendicular to the hyperplane at O lies in both planes.

Theorem 2. If a plane is perpendicular to a hyperplane, any line in the hyperplane perpendicular to their intersection is perpendicular to the plane, and any line perpendicular to the plane through a point of the hyperplane lies entirely in the hyperplane. (Fig. 34.)

Case 1. Given: A plane $\alpha \perp$ to a hyperplane X, with α intersecting X in a line c, and any line b in X $\perp$ to their line of intersection c.

To Prove: b is $\perp$ to α.

Proof: Let Q be the point at which the line b is $\perp$ to c. Let γ be the plane in X $\perp$ to c at Q. b, lying in the hyperplane X and $\perp$ to the line of intersection c, lies also in the plane γ which in X is $\perp$ to c at the same point Q. But γ is $\perp$ to α

(Art. 23, Th. 2). Therefore b is $\perp$ to α. (Q.E.D)

Case 2. Given a plane α $\perp$ to a hyperplane X, with α intersecting X in a line c, and any line b $\perp$ to α through a point P of X. - fig-35 -

To Prove: b lies entirely in X.

Proof: Let γ be the plane $\underline{Y}$ to α and passing through P of X. Since b is given as passing through a point P of the hyperplane X and $\perp$ to α, it lies in the $\underline{Y}$ plane which passes through the same point P, and therefore in X (Art. 23, Th. 2). Therefore b lies entirely in X. (Q.E.D)

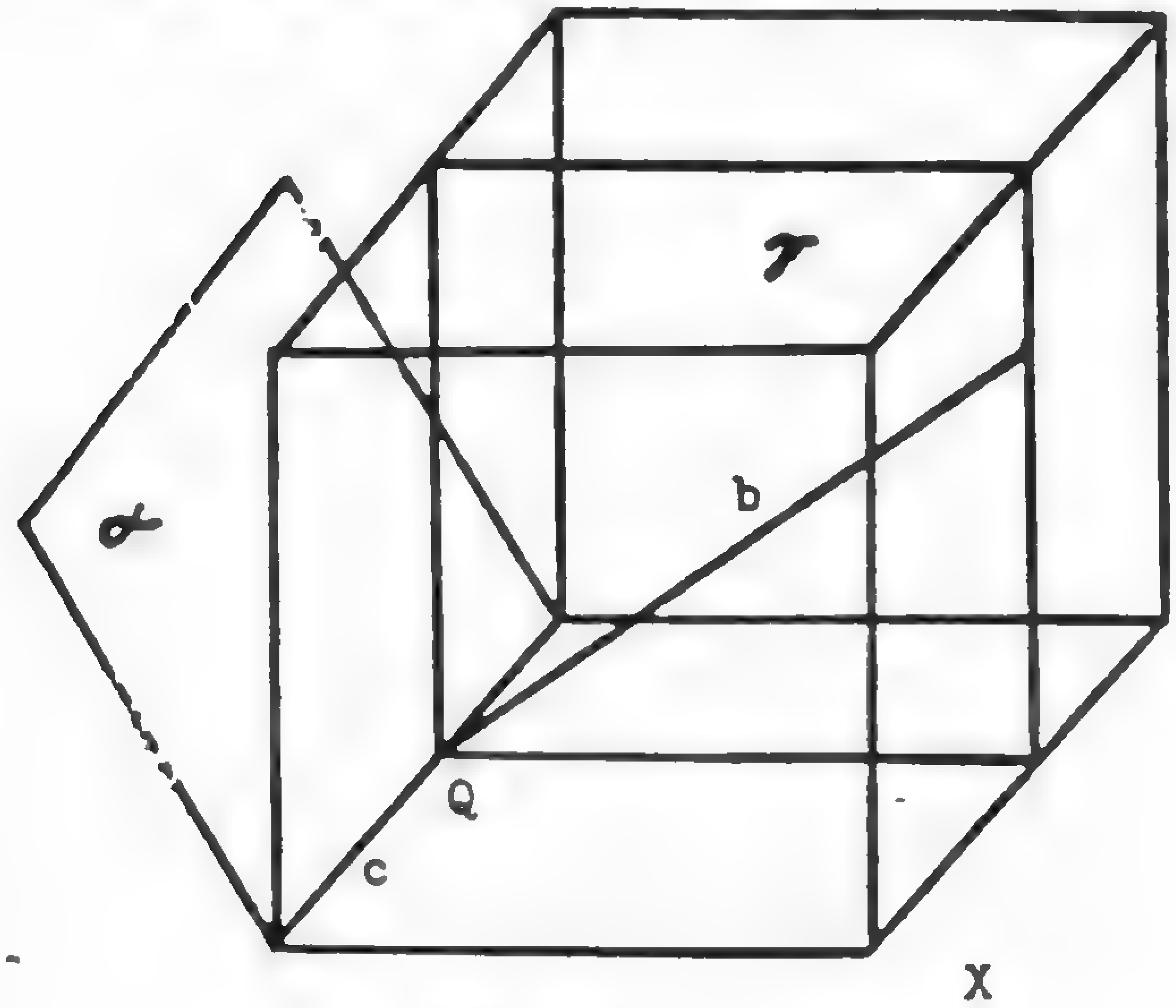


Fig. 34.

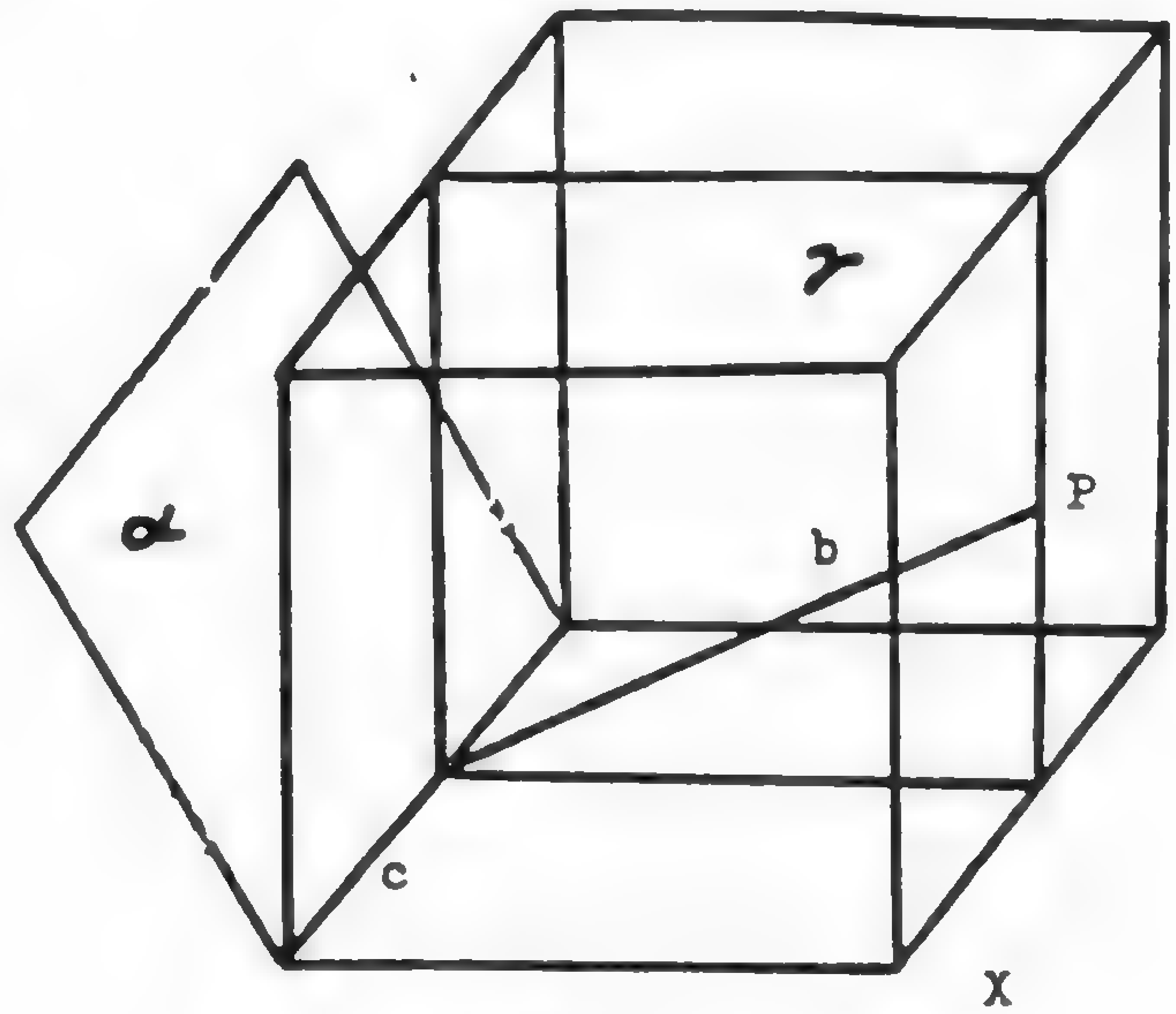


Fig. 35.

25. THROUGH ANY LINE 1 PLANE PERPENDICULAR TO A GIVEN HYPERPLANE, OR 1 HYPERPLANE PERPENDICULAR TO A GIVEN PLANE.

Theorem 1. If a line is perpendicular to a hyperplane, any plane passing through the line is perpendicular to the hyperplane. (see Fig. 32.)

Given: A line b $\perp$ to a hyperplane X, and any plane α passing through the line b.

To Prove: α is $\perp$ to X.

Proof: Let Q be the point where the line b meets the hyperplane X. Let γ be the plane $\underline{Y}$ to α at Q. Now the plane γ $\underline{Y}$ to α at the point Q where b meets X is $\perp$ to the line b, and therefore lies in the hyperplane X (Art. 12, Th. 3). Therefore α is $\perp$ to X. (Q.E.D)

Theorem 2. Through a line not perpendicular to a hyperplane passes 1 and only 1 plane perpendicular to the hyperplane. (Fig. 36.)

Given: A line b not $\perp$ to a hyperplane X.

To Prove: 1 and only 1 plane passing through the line b can be $\perp$ to X.

Proof: Let α be the plane $\perp$ to X along the projection of the line b upon X, then α contains the line b (Art. 24, Th. 1); and we cannot have 2 $\perp$ planes containing the line b, for that would make the line itself $\perp$ to X, which is contrary to hypothesis (same reference and cor.). Therefore 1 and only 1 plane passing through the line b can be $\perp$ to X. (Q.E.D)

Theorem 3. Through a line not lying in a plane absolutely-perpendicular to a given plane, passes 1 and only 1 hyperplane perpendicular to the given plane. (Fig. 37.)

Given: A line b not lying in a plane $\underline{Y}$ to a plane α.

To Prove: 1 and only 1 hyperplane passing through the line b can be $\perp$ to α.

Proof: Let X be the hyperplane $\perp$ to α along the projection of the line b upon the plane α, then X contains the line b (Art. 24, Th. 2); and we cannot have 2 $\perp$

hyperplanes containing the line b, for that would put the line itself into a plane $\underline{V}$ to the plane α, which is contrary to hypothesis (Art. 23, Th. 4). Therefore 1 and only 1 hyperplane passing through the line b can be $\perp$ to α. (Q.E.D)

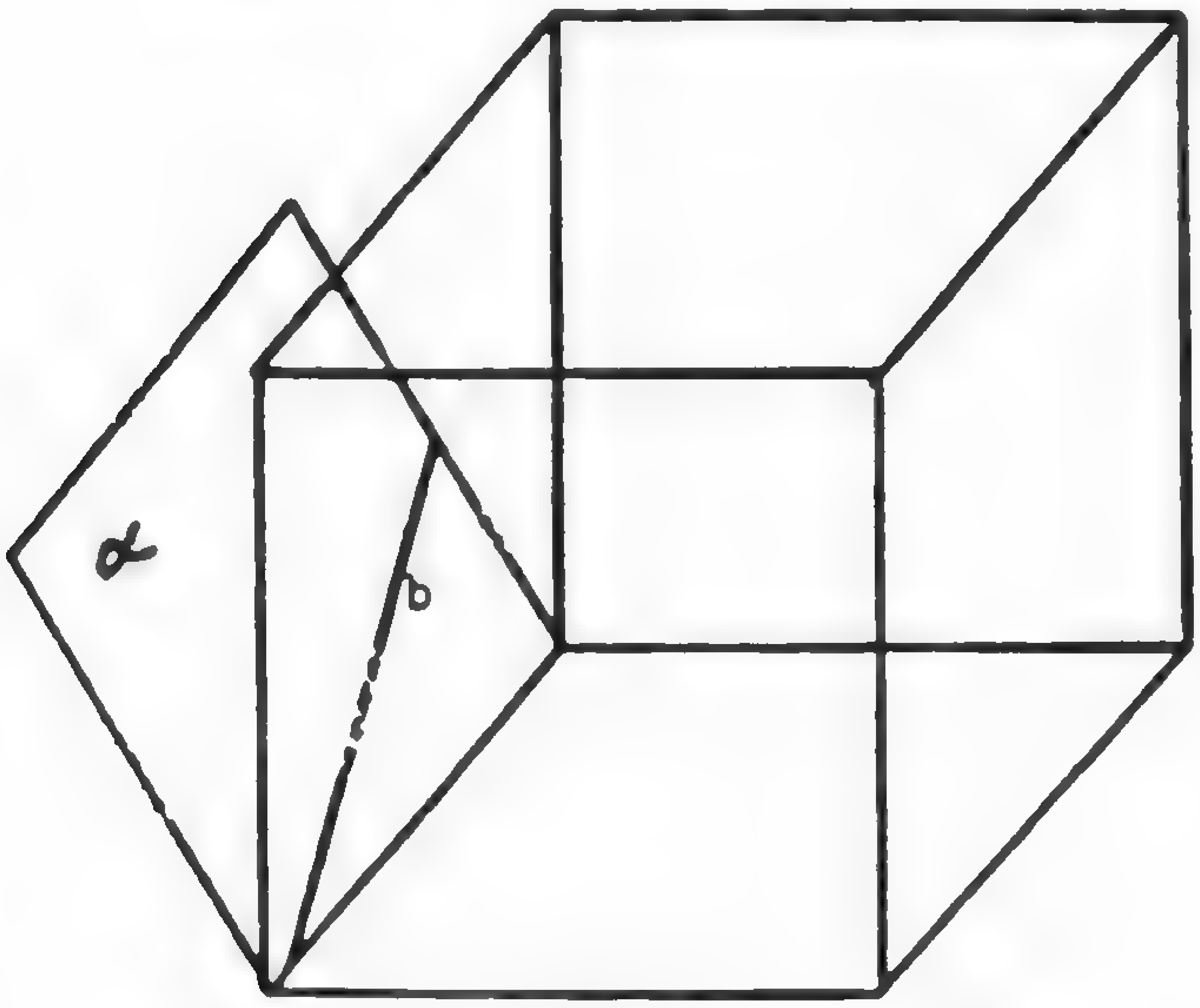


Fig. 36.

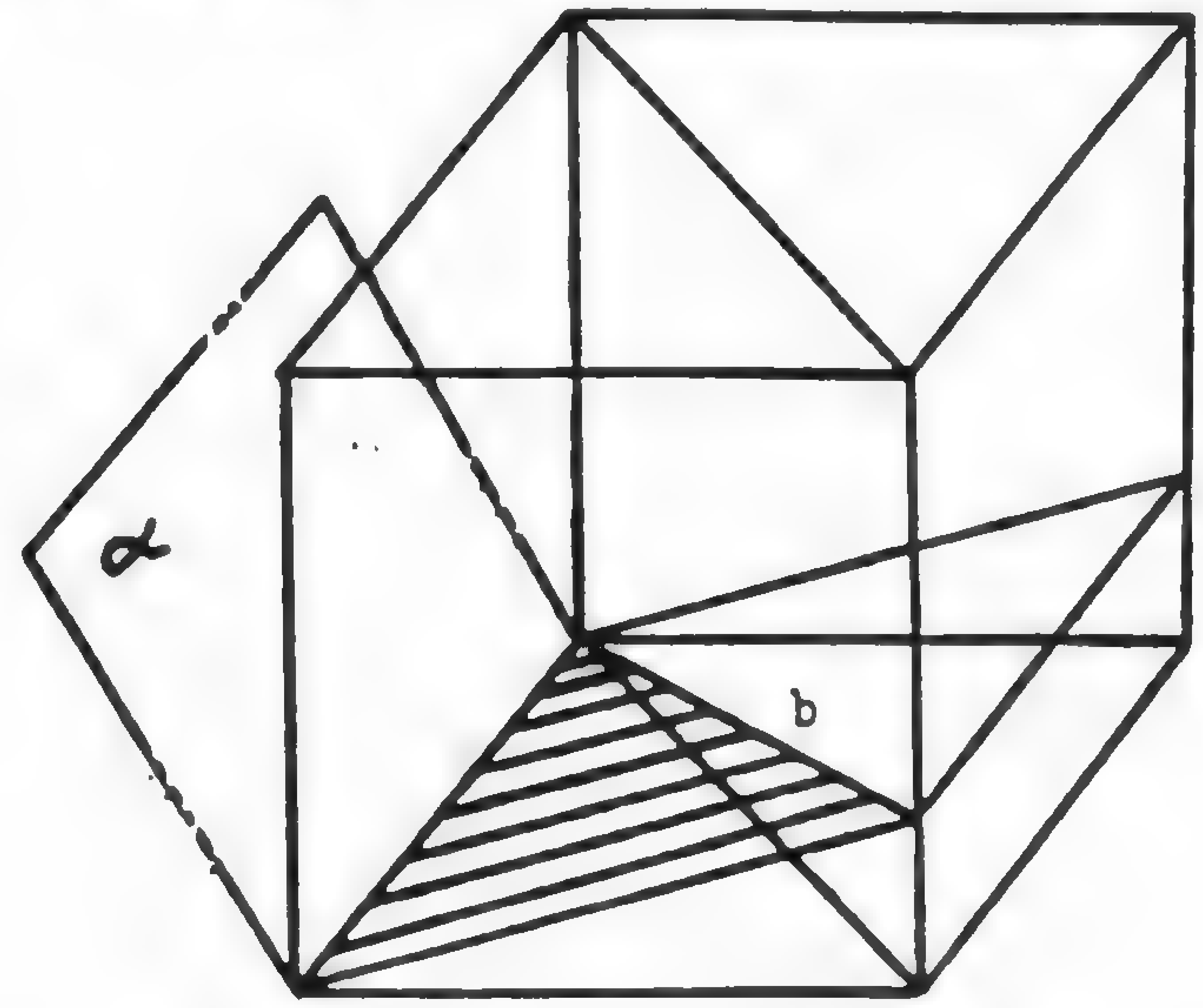


Fig. 37.

26. PLANES WITH LINEAR-ELEMENTS ALL PERPENDICULAR TO A HYPERPLANE.

Theorem. Given 2 planes not in 1 hyperplane, if any 2 of their linear-elements have a common-perpendicular line, they all have a common-perpendicular hyperplane, to which the 2 given planes are also perpendicular. (Fig. 38.)

Given: 2 planes α and β not in 1 hyperplane, and any 2 linear-elements a and b in α and β respectively, with a and b having a common $\perp$ line c.

To Prove: All the linear-elements of α and β have a common $\perp$ hyperplane, to which the 2 planes α and β are also $\perp$.

Proof: Let c be the common $\perp$ line to a and b, with a meeting c at the point P, and b meeting c at the point Q. Let X be the hyperplane $\perp$ to 1 of the lines a or b, say, a, then X being $\perp$ to a at P is $\perp$ to the plane of a and b (Art. 25, Th. 1), and therefore $\perp$ to b (Art. 24, Th. 1). X is also $\perp$ to the planes α and β, as well as to every plane containing a or b (Art. 25, Th. 1). Now any other linear-element h in α is the intersection of α with a plane through b, and any other linear-element k in β is the intersection of β with a plane through a. The hyperplane X is therefore $\perp$ to all of these elements (Art. 24, Th. 1 and Cor.).

If 2 elements in 1 of the given planes are given as having a common $\perp$ line, then the $\perp$ hyperplane X is $\perp$ to the plane in which they lie and to every plane containing either one of them: that is, if a and h are any 2 elements of α having a common $\perp$ line d, then X is $\perp$ to the plane α in which they lie and to every plane containing either a or h. But any element b of β is the intersection of 2 planes through the 2 elements a and h, and so is therefore, as before, $\perp$ to X. Thus X is $\perp$ to elements in both planes and so $\perp$ to all elements and to the 2 planes α and β. Therefore all the linear-elements of α and β have a common $\perp$ hyperplane, to which the 2 planes α and β are also $\perp$. (Q.E.D)

V. HYPERPLANE-ANGLES

27. DEFINITION. INTERIOR. PLANE-ANGLES. A HYPERPLANE-ANGLE consists of 2 $\frac{1}{2}$-hyperplanes having a common-face but not themselves parts of the same hyperplane, together with the common-face.

The common-face is the FACE OF THE HYPERPLANE-ANGLE, and the 2 $\frac{1}{2}$-hyperplanes are the CELLS.

The INTERIOR OF A HYPERPLANE-ANGLE consists of the interiors of all segments whose points are points of the hyperplane-angle, except those segments whose interiors also lie in the hyperplane-angle; that is, the interior of a hyperplane-angle consists of the

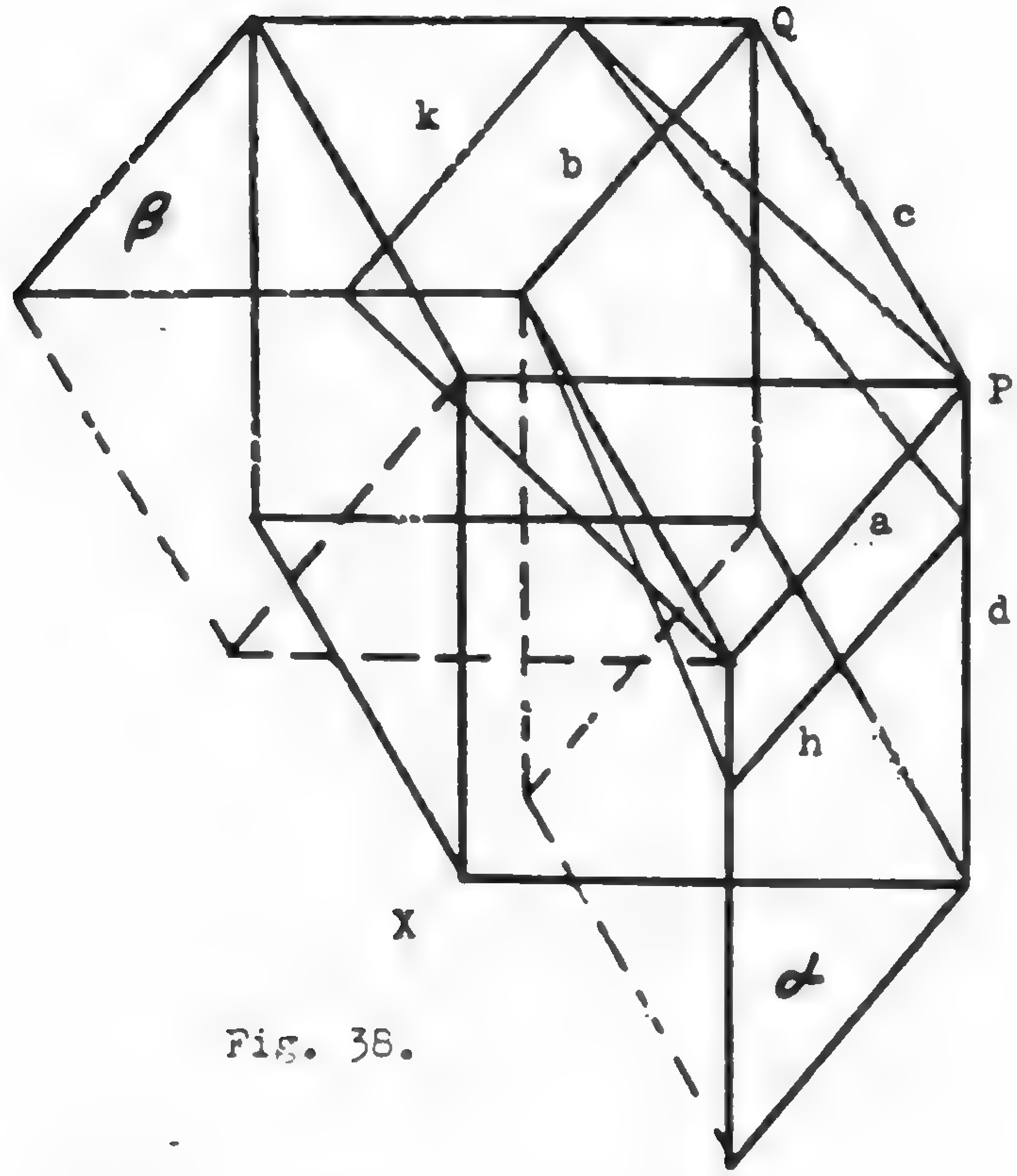


Fig. 38.

interiors of those segments which have a point in each cell. The hyperplane-angle divides the rest of hyperspace into 2 regions, interior and exterior to the hyperplane-angle. Each cell of a hyperplane-angle lies on side of the hyperplane of the other cell; and the portion of hyperspace which lies on that side of the hyperplane of each cell on which lies the other cell, lies BETWEEN the 2 cells and constitutes the interior of the hyperplane-angle.

2 ½-lines drawn from a point O in the face of a hyperplane-angle, 1 in each cell, and each perpendicular to the face, are the sides of an angle which is called the PLANE-ANGLE AT O OF THE HYPERPLANE-ANGLE.

In Fig. 39, we have the representation of a hyperplane-angle. The hyperplane-angle is designated by D-OBC-A. The ½-hyperplane cells of the hyperplane-angle D-OBC-A are designated as OBC-A and OBC-D, and their common-face as OBC. The 2 ½-lines OA and OD drawn from a point O in the face OBC of a hyperplane-angle D-OBC-A, with OA in cell OBC-A, OD in cell OBC-D, and with OA and OD $\perp$ to the face OBC, are the sides of an angle AOD which is called the plane-angle at O of the hyperplane-angle D-OBC-A.

Theorem 1. The plane absolutely-perpendicular at a point O to the face of a hyperplane-angle, intersects the hyperplane-angle in the plane-angle at O.

Theorem 2. A hyperplane perpendicular to the face of a hyperplane-angle intersects the hyperplane-angle in a dihedral-angle which at any point of its edge has the same plane-angle as the hyperplane-angle. (Fig. 40.)

Given: A hyperplane X $\perp$ to the face OBC of a hyperplane-angle D-OBC-A, with X intersecting OBC in a line OC.

To Prove: X intersects D-OBC-A in a dihedral-angle D-OC-A which at any point O of its edge OC has the same plane-angle AOD as D-OBC-A.

Proof: The intersection consists of the ½-planes OC-A and OC-D with a common-edge OC lying in the face OBC of the hyperplane-angle D-OBC-A. The ½-planes OC-A and OC-D with the common-edge OC form a dihedral-angle D-OC-A. The plane α $\perp$ to the face OBC at any point O of the edge OC, lies in the $\perp$ hyperplane X, and in X is $\perp$ to OC (Art. 23, Th. 2). The plane α intersects the dihedral-angle D-OC-A in the same angle in which it intersects D-OBC-A—an angle AOD which is, therefore, the plane-angle at O of both. Therefore X intersects D-OBC-A in a dihedral-angle D-OC-A which at any point O of its edge OC has the same plane-angle AOD as D-OBC-A. (Q.E.D)

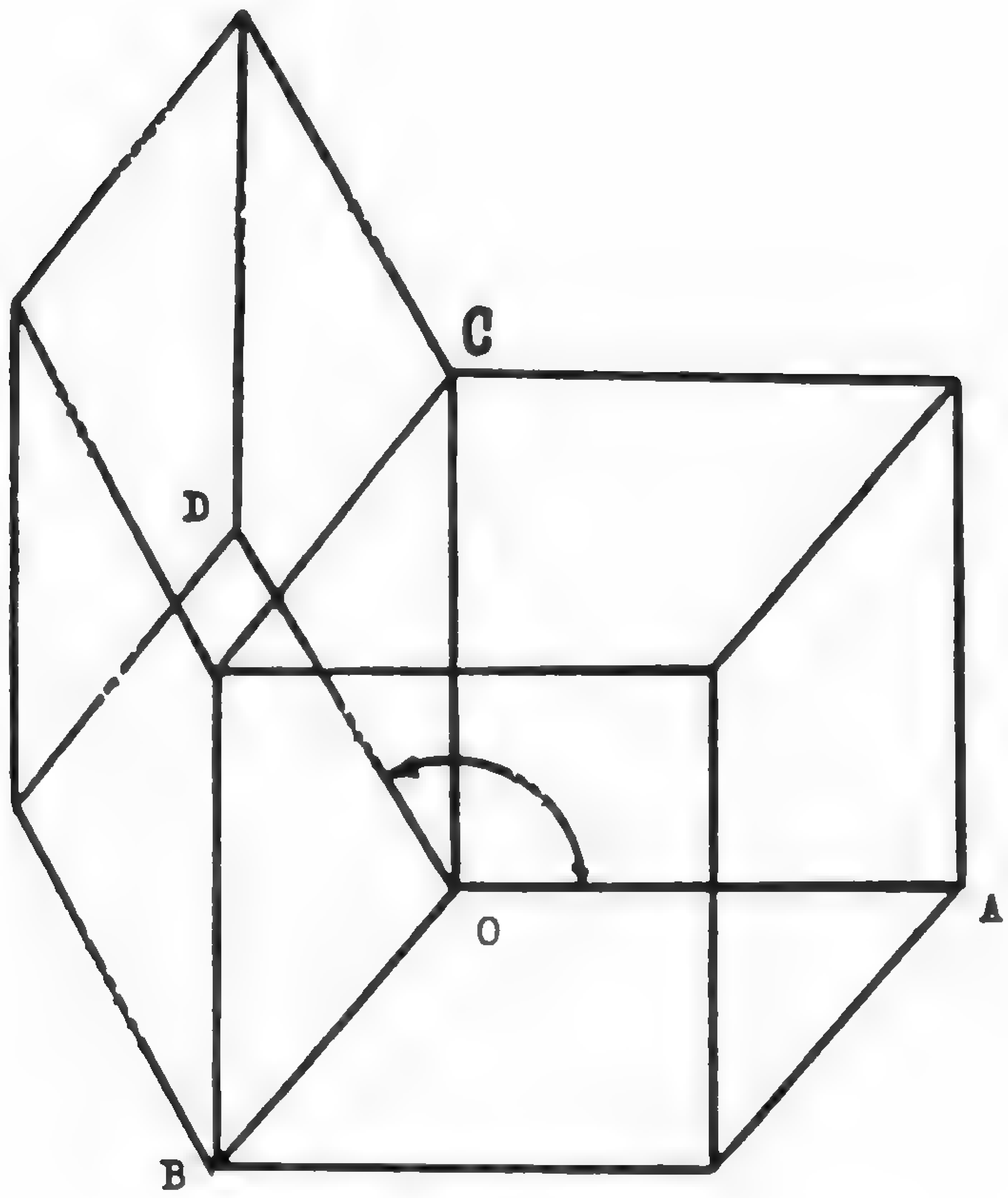


Fig. 39.

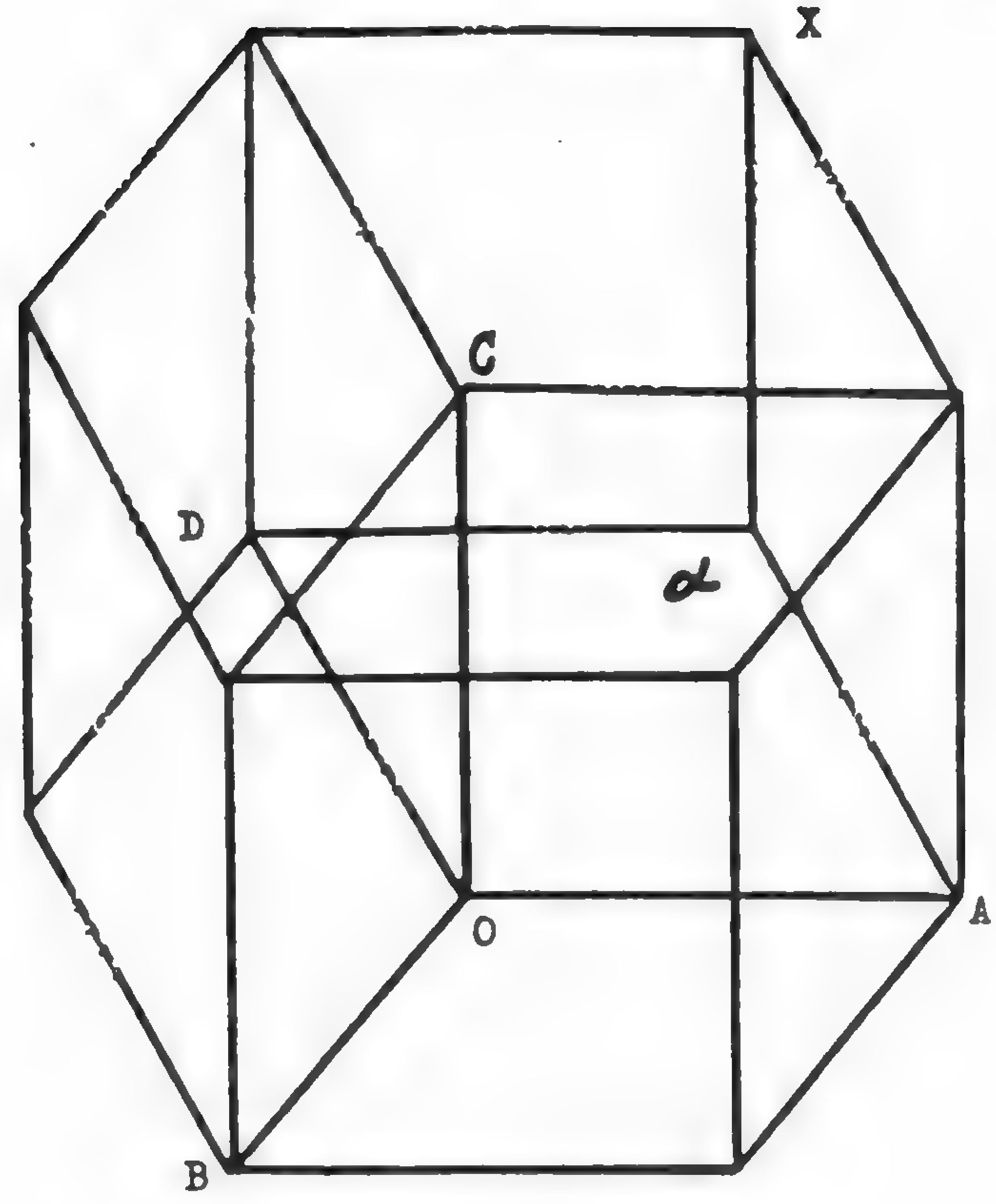


Fig. 40.

Theorem 3. 2 hyperplane-angles are congruent if a plane-angle of one is equal to a plane-angle of the other. (Fig. 41a and Fig. 41a'.)

Given: 2 hyperplane-angles D-OBC-A and D'-O'B'C'-A', with the plane-angle AOD at the point O of D-OBC-A equal to the plane-angle A'O'D' at the point O' of D'-O'B'C'-A'.

To Prove: D-OBC-A is congruent to D'-O'B'C'-A'.

Proof: By superposition.
Place plane-angle AOD on plane-angle A'O'D' so that side OA coincides with side O'A' and side OD coincides with side O'D'. Then face OBC will coincide with face O'B'C', and OBC $\underline{V}$ to the plane α of the plane-angle AOD at the point O will coincide with O'B'C' $\underline{V}$ to the plane α' of the plane-angle A'O'D' at the point O'. Now OA in cell OBC-A coincides with O'A' in cell O'B'C'-A' and OD in cell OBC-D coincides with O'D' in cell O'B'C'-D'. Therefore cell OBC-A will coincide with cell O'B'C'-A' and cell OBC-D will coincide with cell O'B'C'-D', and the 2 hyperplane-angles D-OBC-A and D'-O'B'C'-A' coincide throughout and are congruent (Art. 1, Th. 2(1) and Th. 3). Therefore D-OBC-A is congruent to D'-O'B'C'-A'. (Q.E.D)

If cell OBC-A coincides with cell O'B'C'-A', with the points O and O' coinciding; and cell OBC-D coinciding with cell O'B'C'-D', with the points D and D' coinciding. Then D-OBC-A and D'-O'B'C'-A' will coincide throughout. We then have the plane α coinciding with the plane α' and the face OBC coinciding with the face O'B'C' such that α $\underline{V}$ to OBC at O will coincide with α' $\underline{V}$ to O'B'C' at O', and the plane-angle AOD will coincide with the plane-angle A'O'D', with side OA coinciding with side O'A' and side OD coinciding with side O'D'.

Theorem 4. The plane-angle of a hyperplane-angle is the same at all points of the face. (Fig. 42.)

Given: A plane-angle AOD at a point O of the face OBC of a hyperplane-angle D-OBC-A.

To Prove: The plane-angle AOD of the hyperplane-angle D-OBC-A is the same at all points of the face OBC of D-OBC-A.

Proof: Let α be the plane $\underline{V}$ to the face OBC at the point O which contains the given plane-angle AOD of the hyperplane-angle D-OBC-A. At any other point, say C, pass a plane β $\underline{V}$ to OBC at C. The 2 planes α and β $\underline{V}$ to the face OBC at points O and C respectively,

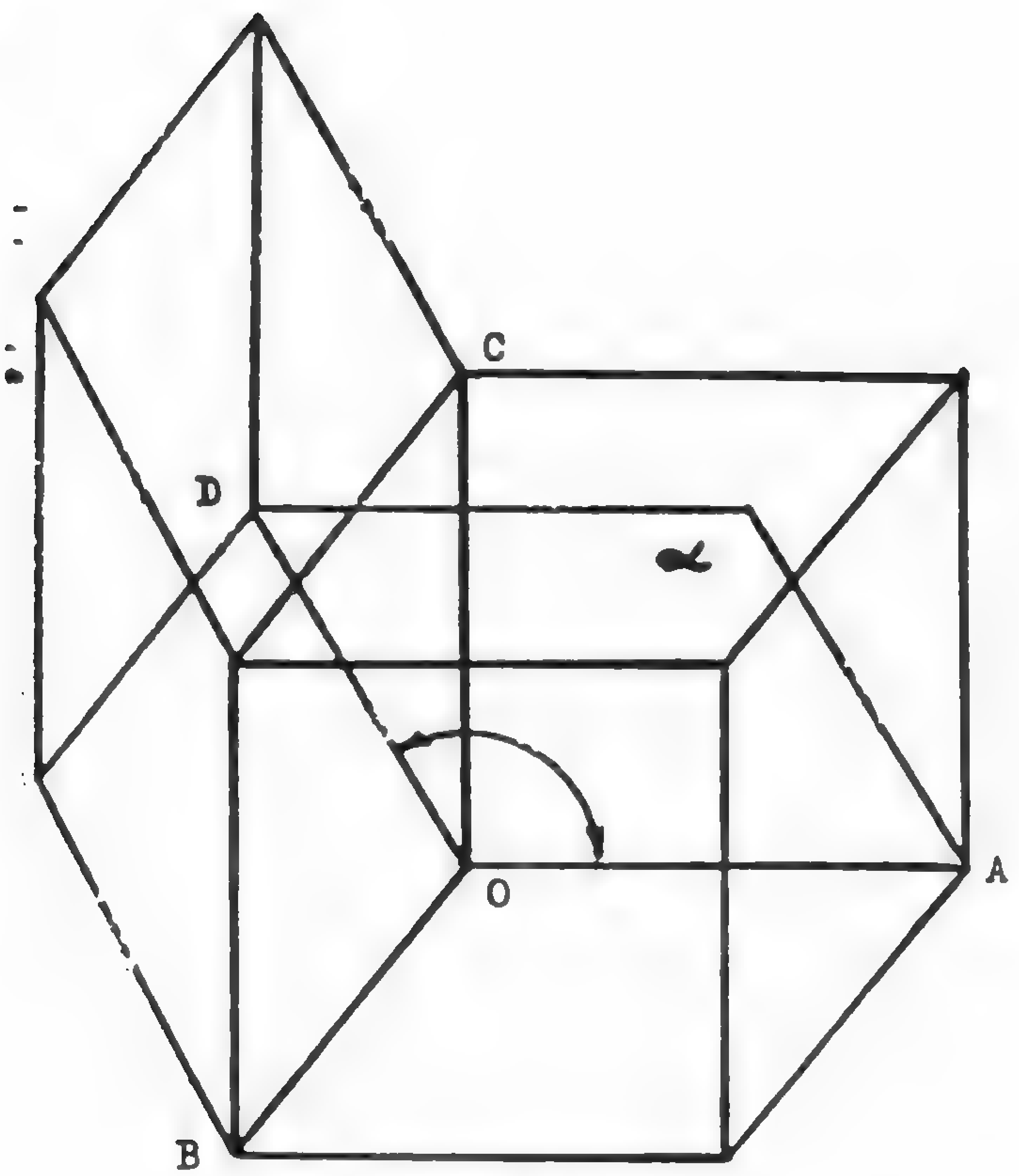


Fig. 41a.

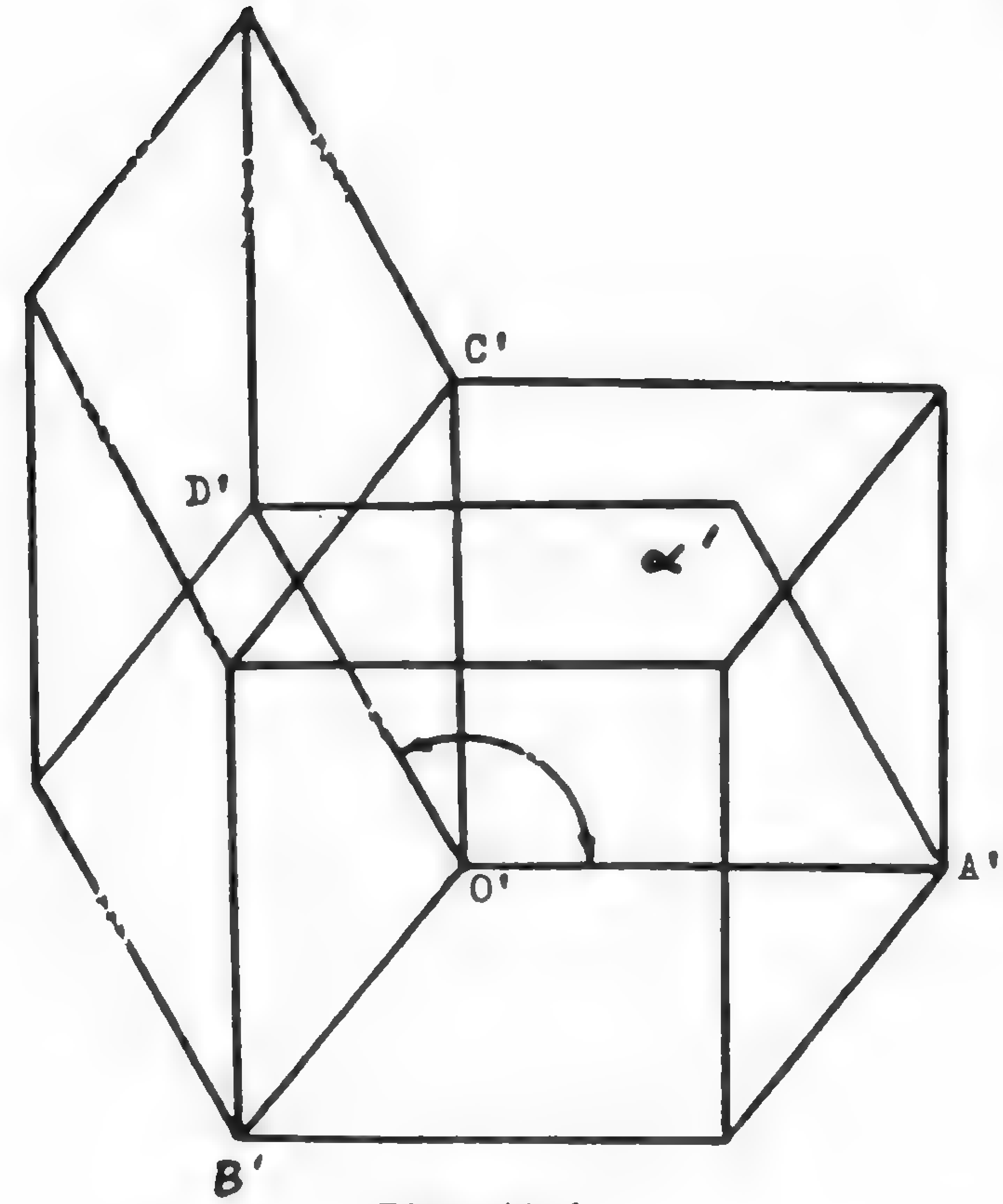


Fig. 41a'.

lie in a $\perp$ hyperplane (Art. 18, Th.), in which the corresponding angles are 2 plane-angles of the same dihedral-angle; that is, if β is the plane of the plane-angle ECF, and given the plane α of the plane-angle AOD, then these 2 plane-angles are the corresponding plane-angles of the same dihedral-angle A-OC-D. Therefore plane-angle ECF is = to plane-angle AOD. Therefore the plane-angle AOD of the hyperplane-angle D-OBC-A is the same at all points of the face OBC of D-OBC-A. (Q.E.D)

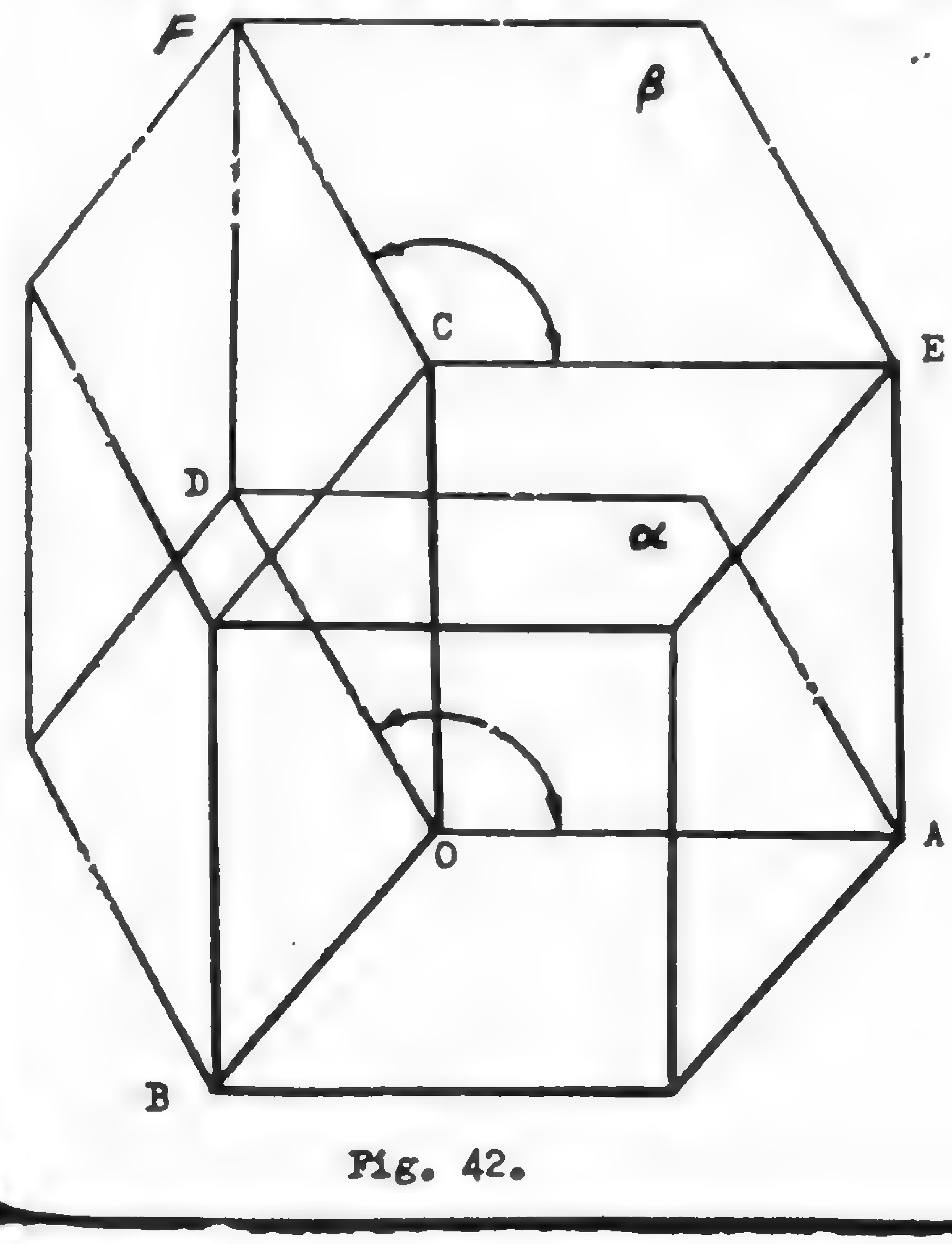


Fig. 42.

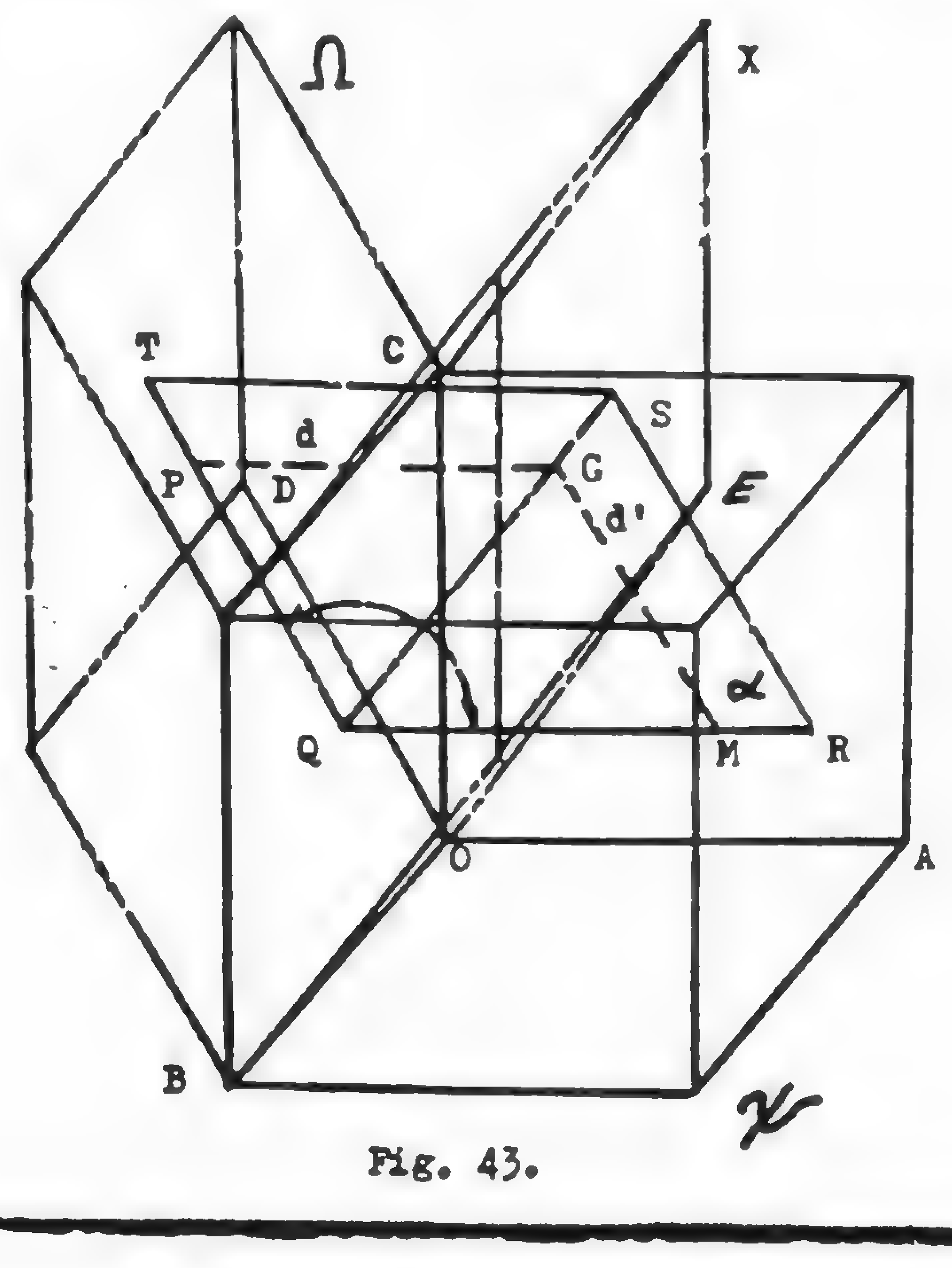


Fig. 43.

Corollary. 2 hyperplane-angles which are congruent in any position will always coincide as soon as they have a common-cell and the other cells lie on the same-side of the hyperplane of this common-cell.

28. THE HYPERPLANE-ANGLE AS A MAGNITUDE. KINDS OF HYPERPLANE-ANGLES.

SUPPLEMENTARY HYPERPLANE-ANGLES are those which can be placed so as to have 1 cell in common while their other cells are opposite ½-hyperplanes. Each of them is then the SUPPLEMENT of the other. A RIGHT HYPERPLANE-ANGLE is one which is congruent to its supplement. The hyperplanes of the cells of a right-hyperplane-angle are said to be PERPENDICULAR.

If 2 hyperplane-angles have a common-face and the cells of one lie in the interior of the other, or if they have 1 cell in common while 1 cell of one lies in the interior of the other, then the interior of one hyperplane-angle is a part of the interior of the other. We shall speak of one hyperplane-angle as a PART OF THE OTHER and as LESS IN MAGNITUDE.

Let 2 hyperplane-angles be placed so as to have 1 cell in common while their other cells and their interiors lie on opposite-sides of the hyperplane of this common-cell. Then, if either hyperplane-angle is less than the supplement of the other, these 2 cells and the common-face will form a hyperplane-angle which we can call the SUM OF THE GIVEN HYPERPLANE-ANGLES.

Theorem 1. 2 supplementary-hyperplane-angles have supplementary-plane-angles; a right-hyperplane-angle has a right-plane-angle; the smaller of 2 unequal hyperplane-angles has the smaller plane-angle; and the plane-angle of the sum of 2 hyperplane-angles is the sum of their plane-angles.

Theorem 2. If we divide the plane-angle of a hyperplane-angle into any number of equal parts, the lines of division will determine additional cells by means of which the given hyperplane-angle is divided into the same number of equal parts; in particular, given a hyperplane-angle α, and other hyperplane-angle may be divided into a sufficient number of equal parts so that 1 of these parts shall be less than α.

We can build-up a theory of the measurement of hyperplane-angles, taking any particular one as the unit of measure.

29. HYPERPLANE-ANGLES MEASURED BY THEIR PLANE-ANGLES.

Theorem. 2 hyperplane-angles are in the same ratio as their plane-angles, and the hyperplane-angle may be measured by its plane-angle.

Proof: If we divide the plane-angle of a hyperplane-angle into some number of equal parts, the ½-lines of division, taken with the face of the hyperplane-angle, will determine ½-hyperplanes which divide the interior of the hyperplane-angle into the same number of equal parts. We can therefore prove in the usual way that hyperplane-angles are proportional to their plane-angles, 1st when the plane-angles are commensurable, 2nd when the plane-angles are incommensurable using the method of limits or some equivalent method.

Now a right-hyperplane-angle has a right-plane-angle (Art. 28, Th. 1). Therefore, the measure of the hyperplane-angle in terms of a right-hyperplane-angle is always the same as the measure of its plane-angle in terms of a right-angle.

30. THE BISECTING ½ HYPERPLANE.

Theorem 1. The ½-hyperplane bisecting a hyperplane-angle is the locus of points in the interior of the hyperplane-angle equidistant from the hyperplanes of its cells. (Fig. 43.)

Given: The ½-hyperplane OBC-E bisecting a hyperplane-angle D-OBC-A.

To Prove: The bisecting ½-hyperplane OBC-E is the locus of points in the interior of the hyperplane-angle D-OBC-A equidistant from the hyperplanes of its cells.

Proof: Through any point G lying in the interior of the hyperplane-angle D-OBC-A we can pass a plane $\perp$ to the face OBC of D-OBC-A (Art. 16, Th. 2). Let α be the plane passing through G and $\perp$ to OBC at the point Q. The plane α will intersect the cells of the hyperplane-angle D-OBC-A in the sides of a plane-angle and the bisecting ½-hyperplane OBC-E in the ½-line which bisects the plane-angle. Let the ½-line QR be the intersection

of α with the cell OBC-A and the $\frac{1}{2}$-line QT be the intersection of α with the cell OBC-D, then the $\frac{1}{2}$-lines QR and QT are the sides of a plane-angle RQT. Let the $\frac{1}{2}$-line QS which bisects the plane-angle RQT be the intersection of α with the bisecting $\frac{1}{2}$-hyperplane OBC-E. Now α is $\perp$ to the hyperplane of the cells OBC-A and OBC-D of the hyperplane-angle D-OBC-A (Art. 23, Def.), and the distances of the point G from the hyperplanes of the cells are the distances of G from the sides QR and QT of the plane-angle RQT (Art. 24, Th. 1). If, then, the point G is in the $\frac{1}{2}$-hyperplane OBC-E bisecting the hyperplane-angle D-OBC-A, it is in the $\frac{1}{2}$-line QS bisecting the plane-angle RQT, and these distances are equal: that is, if M is the projection of G on QR, with GM = d', and P is the projection of G on QT, with GP = d, then d = d'. If d = d', the point G is in the $\frac{1}{2}$-line QS bisecting the plane-angle RQT, and therefore in the $\frac{1}{2}$-hyperplane OBC-E bisecting the hyperplane-angle D-OBC-A. That is, the bisecting $\frac{1}{2}$-hyperplane OBC-E is the locus of points in the interior of the hyperplane-angle D-OBC-A equidistant from the hyperplanes of its cells. Therefore the statement above is proved. (Q.E.D)

Theorem 2. The distances of a point in one cell of a hyperplane-angle from the bisecting $\frac{1}{2}$-hyperplane, is greater than $\frac{1}{2}$ of the distance of the point from the plane of the other cell.

The proof is like that of the preceding theorem, by passing a plane through the given point absolutely-perpendicular to the face of the hyperplane-angle.

31. PERPENDICULAR-HYPERPLANES. LINES LYING IN ONE AND PERPENDICULAR TO THE OTHER.

Theorem 1. If 2 hyperplanes are perpendicular, any line in one perpendicular to their intersection is perpendicular to the other, and any line through a point of one perpendicular to the other lies entirely in the first. (Fig. 47)

Given: 2 hyperplanes X and γ intersecting in a plane α, with X $\perp$ to γ, and any line a in X $\perp$ to γ. (Case 1.)

To Prove: a is $\perp$ to γ.

Proof: Let O be the point where the line a is $\perp$ to α. The line a lies along one side of a plane-angle of each of the 4 right-hyperplane-angles whose cells lie in the 2 hyperplanes X and γ, and, since these plane-angles are right-angles, the line a is $\perp$ to that line of γ along which lie their other sides. Let b be that line at O along which lie the other sides of the plane-angle at O of these 4 right-hyperplane-angles. The line a is $\perp$ to the line b in γ as well as to α in which X and γ intersect, and therefore is $\perp$ to γ (Art. 12, Th. 4). Therefore a is $\perp$ to γ. (Q.E.D)

Case 2. The proof of the 2nd part of this theorem is like the corresponding proofs given in Art. 24.

Theorem 2. If a line is perpendicular to a hyperplane, any hyperplane which contains the line is perpendicular to the hyperplane.

Theorem 3. If 3 intersecting hyperplanes with only a line common to all 3 are perpendicular to a given hyperplane, the line of intersection is perpendicular to the given hyperplane.

32. PLANES LYING IN ONE AND PERPENDICULAR TO THE OTHER OF 2 PERPENDICULAR-HYPERPLANES.

Theorem 1. If 2 hyperplanes are perpendicular, any plane in one, perpendicular to their intersection, is perpendicular to the other, and any plane through a line of one, perpendicular to the other, lies entirely in the first unless the line itself is perpendicular to the hyperplane of the second. (Fig. 43)

Case 1. Given: 2 hyperplanes X and γ intersecting in a plane α, with X $\perp$ to γ, and any plane β in X, $\perp$ to α.

To Prove: β is $\perp$ to γ.

Proof: Draw a line b in the plane β $\perp$ to α at a point O, and therefore $\perp$ to γ (Th. 1). β is then $\perp$ to γ (Art. 25, Th. 1). Therefore β is $\perp$ to γ. (Q.E.D)

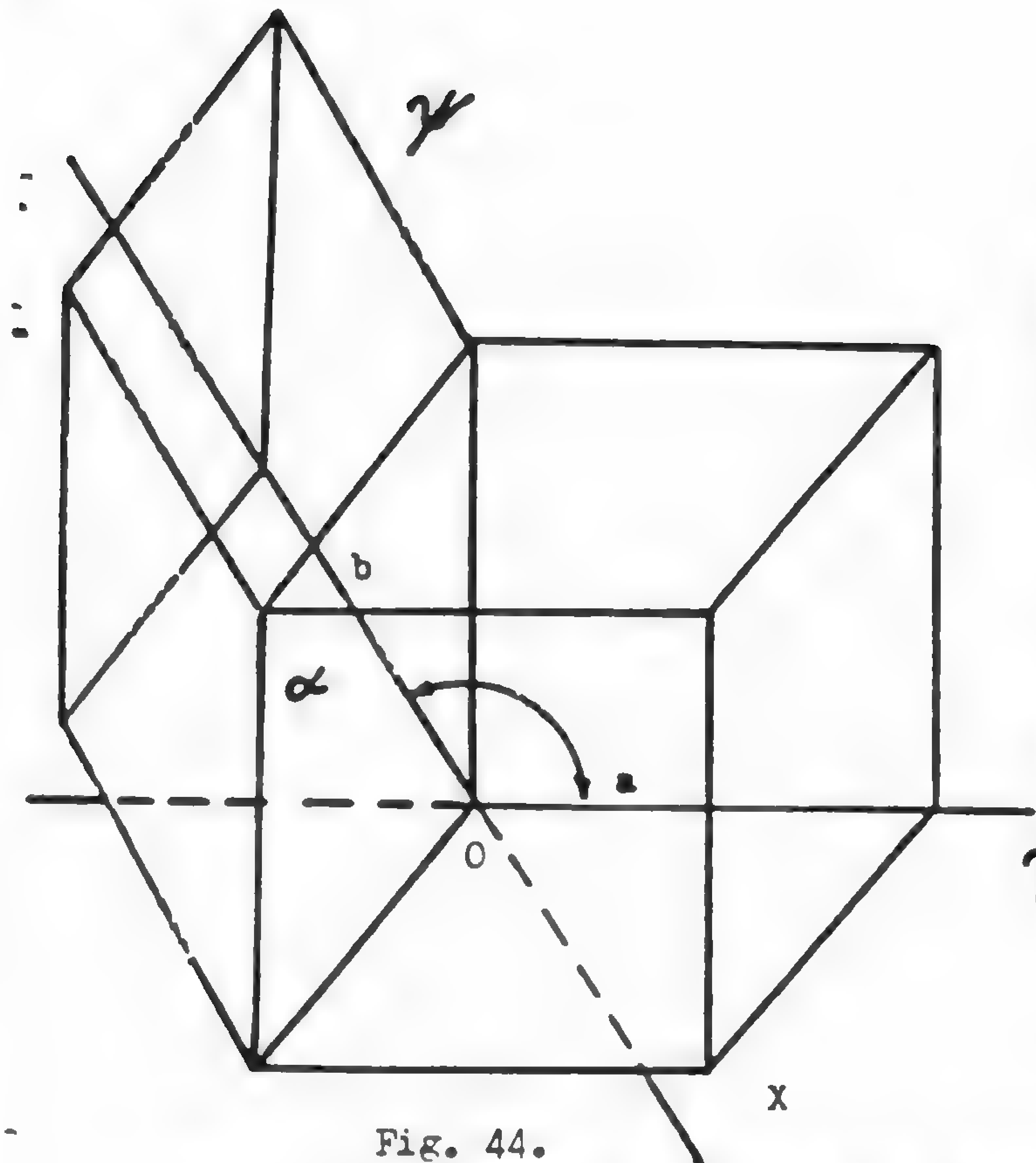


Fig. 44.

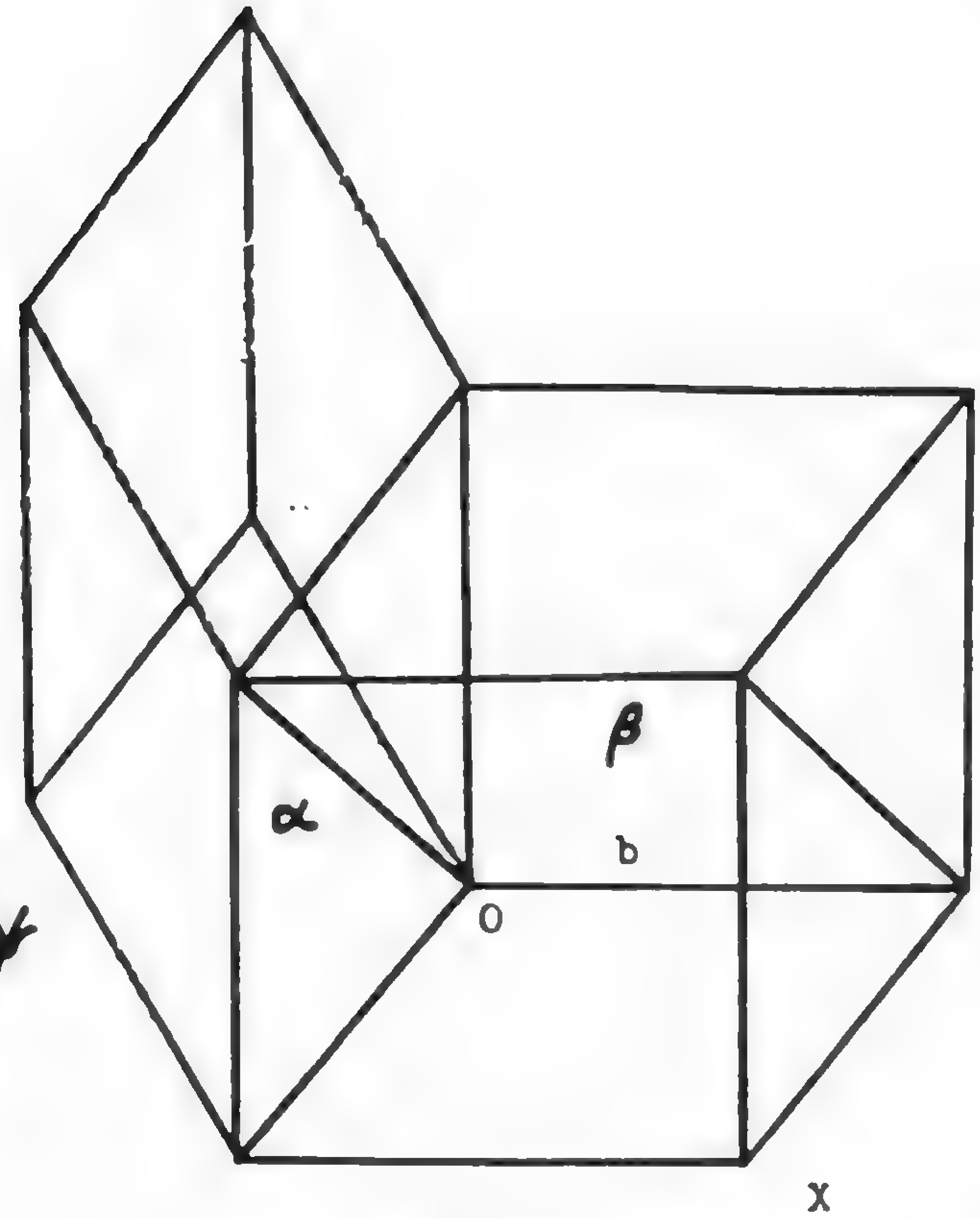


Fig. 45.

Case 2. Given: 2 hyperplanes X and ψ intersecting in a plane α, with X $\perp$ to ψ, and any plane β through a line b of X, $\perp$ to α. (Fig. 46.)

To Prove: β lies entirely in X unless b itself is $\perp$ to ψ.

Proof: Suppose b in X is not $\perp$ to ψ, then we can draw a plane γ through b in X $\perp$ to α, and therefore $\perp$ to ψ, by the 1st part of this theorem. But through a line not $\perp$ to a hyperplane there passes 1 and only 1 plane $\perp$ to the hyperplane (Art. 25, Th. 2). Therefore β must coincide with γ and lie entirely in X. Therefore β lies entirely in X unless b itself is $\perp$ to α. (Q.E.D)

Theorem 2. If a plane is perpendicular to a hyperplane, any hyperplane which contains the plane is perpendicular to the hyperplane. (use Fig. 45.)

Given: A plane β $\perp$ to a hyperplane ψ.

To Prove: Any hyperplane X which contains β is $\perp$ to ψ.

Proof: In the plane β are lines which are $\perp$ to the hyperplane ψ, and any such hyperplane which contains the plane β must contain these lines, and must itself be $\perp$ to ψ (Art. 31, Th. 2). Therefore any hyperplane X which contains β is $\perp$ to ψ. (Q.E.D)

Theorem 3. If 2 intersecting hyperplanes are perpendicular to a given hyperplane, their intersection is also perpendicular to the given hyperplane.

33. PROJECTION OF A PLANE UPON A HYPERPLANE. ANGLE OF A $\frac{1}{2}$-PLANE AND HYPERPLANE.

Theorem 1. The projection of a plane upon a hyperplane is a plane or a part of a plane; it does not lie entirely in 1 line unless the given plane is perpendicular to the hyperplane. (Fig. 47.)

Given: A plane α and a hyperplane X.

To Prove: The projection of α upon X is a plane β or a part of β; it does not lie entirely in 1 line unless the plane α is $\perp$ to X.

Proof: The plane α and any point O of its projection determine a hyperplane ψ which is $\perp$ to X (Art. 31, Th. 2). Let β be the plane of intersection of X and ψ. The lines which projects the points of α upon X are the same as the lines which project the points

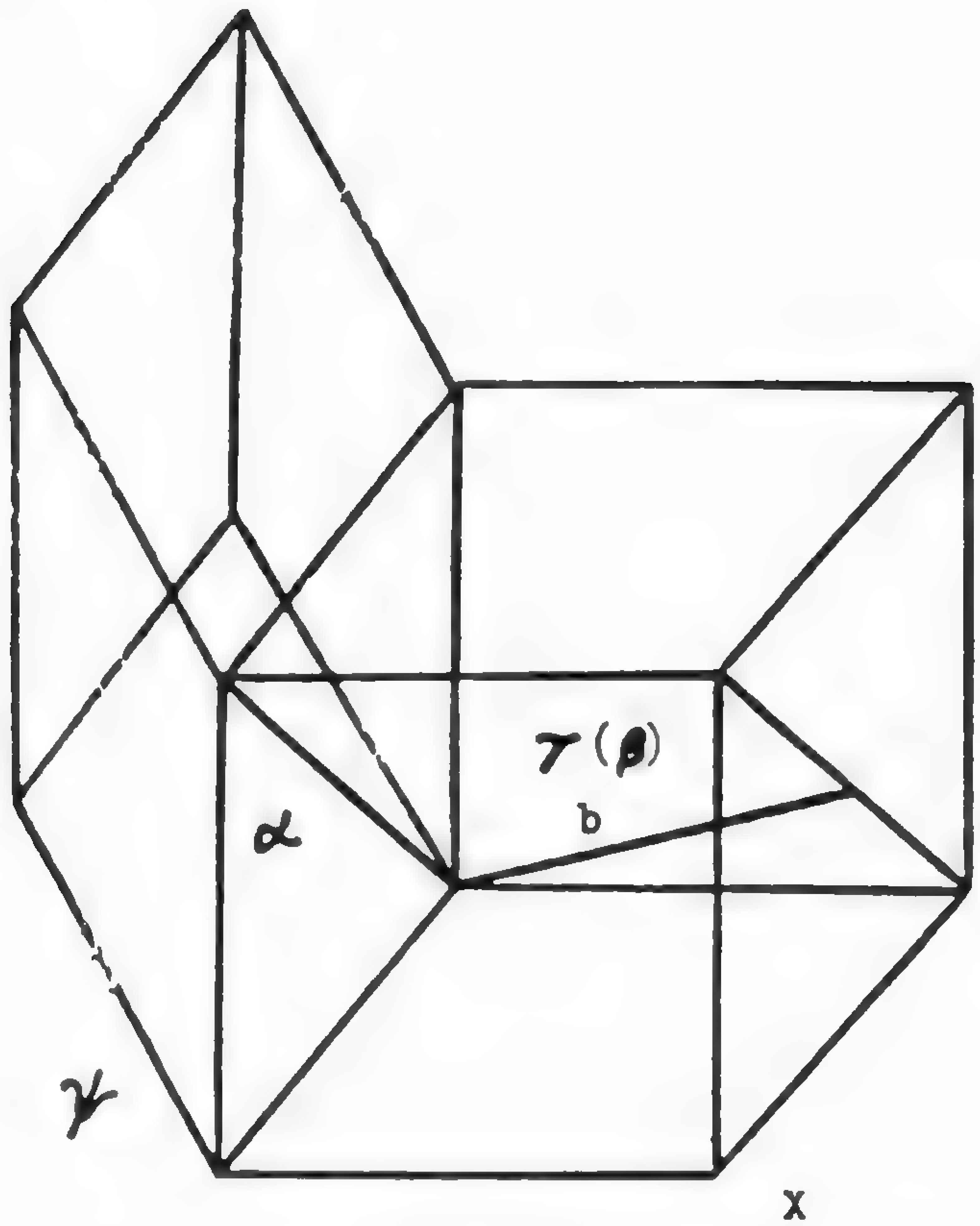


Fig. 46.

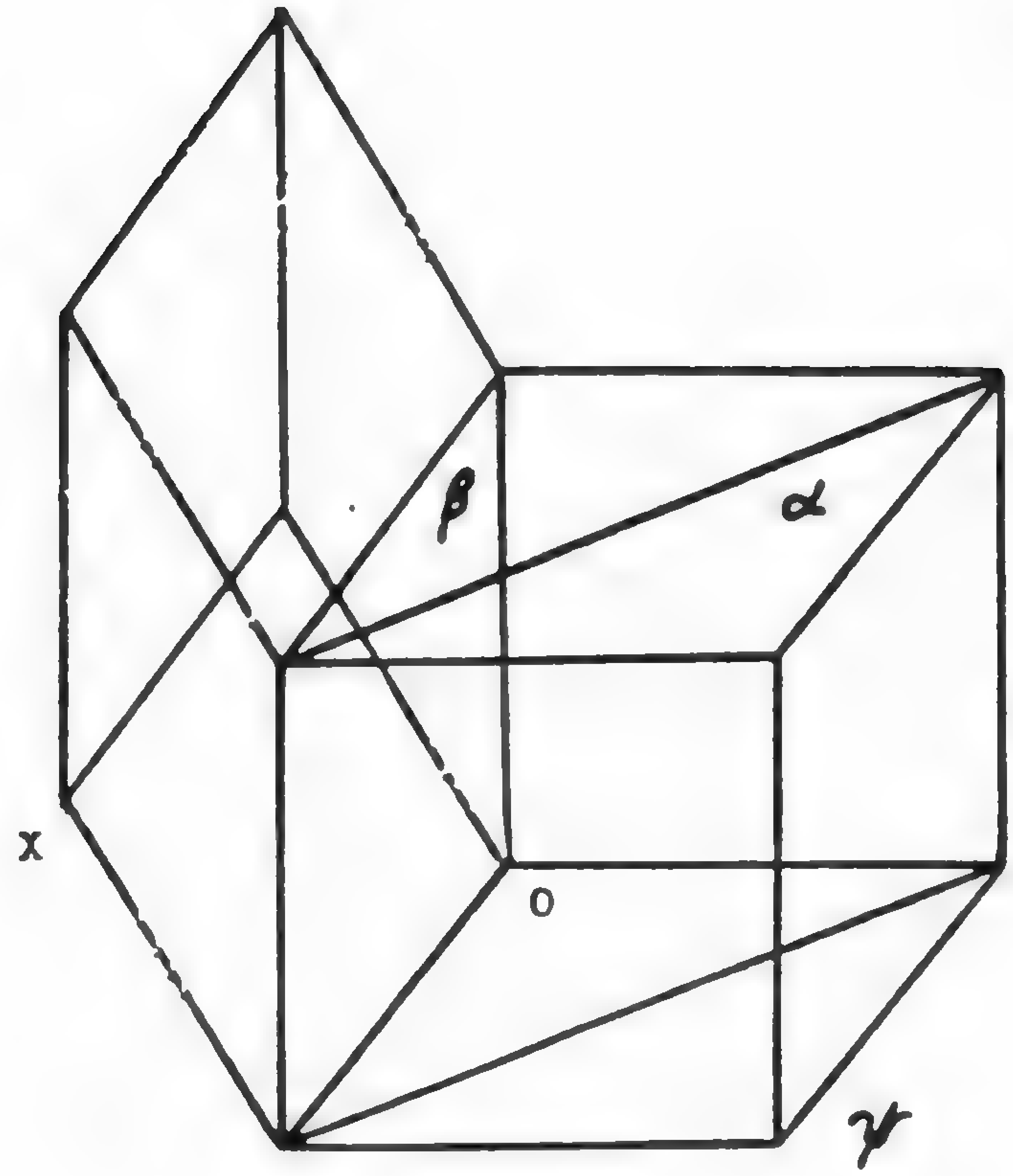


Fig. 47.

of the plane α upon the plane β, the intersection of X and ψ (Art. 31, Th. 1, and Art. 12, Th. 3). The projection upon X is therefore the same as the projection upon the plane of intersection β. Now the plane α is not $\perp$ to β if it is not $\perp$ to X (Art. 32, Th. 1). Therefore the projection is not a line, but is the plane of intersection β itself, or a part of β. Therefore the projection of α upon X is a plane β or a part of β *; it does not lie entirely in 1 line unless the plane α is $\perp$ to X. (Q.E.D)

*It is a single connected convex-part, any point lying between the projection of 2 points being also the projection of a point upon the given plane.

Corollary. When a $\frac{1}{2}$-plane with its edge in a given hyperplane does not lie in the hyperplane and is not perpendicular to it, its projection upon the hyperplane is a $\frac{1}{2}$-plane, or a portion of a $\frac{1}{2}$-plane, having the same edge.

Theorem 2. When a $\frac{1}{2}$-plane with its edge in a given hyperplane does not lie in the hyperplane and is not perpendicular to it, the dihedral-angle which it makes with the $\frac{1}{2}$-plane having the same edge and containing its projection is less than the dihedral-angle which it makes with any other $\frac{1}{2}$-plane of the hyperplane having the same edge. (Fig. 48.)

Given: A $\frac{1}{2}$-plane α with its edge e in a hyperplane X and which does not lie in X and is not $\perp$ to X, with the $\frac{1}{2}$-plane β being the projection of α upon X and having the same edge e as α, and any other $\frac{1}{2}$-plane γ in X with edge e.

To Prove: The dihedral-angle which α makes with β is less than the dihedral-angle which it makes with γ.

Proof: Let ψ be the hyperplane $\perp$ to the common-edge e at a point C. Then ψ will intersect the 3 $\frac{1}{2}$-planes α, β, and γ in 3 $\frac{1}{2}$-lines a, b, and c, respectively, and these $\frac{1}{2}$-lines taken 2 at-a-time, are the sides of plane-angles of the dihedral-angles whose faces are any 2 of the 3 $\frac{1}{2}$-planes.

Now the hyperplane ψ and the hyperplane of α and β contain the $\frac{1}{2}$-lines a and b, and are both $\perp$ to X (the former by Art. 31, Th. 2). Hence they intersect in a plane which contains a and b and is $\perp$ to X (Art. 32, Th. 3), and therefore contains the projection of a upon X (Art. 24, Th. 1). In other words, the $\frac{1}{2}$-line b is the $\frac{1}{2}$-line which contains

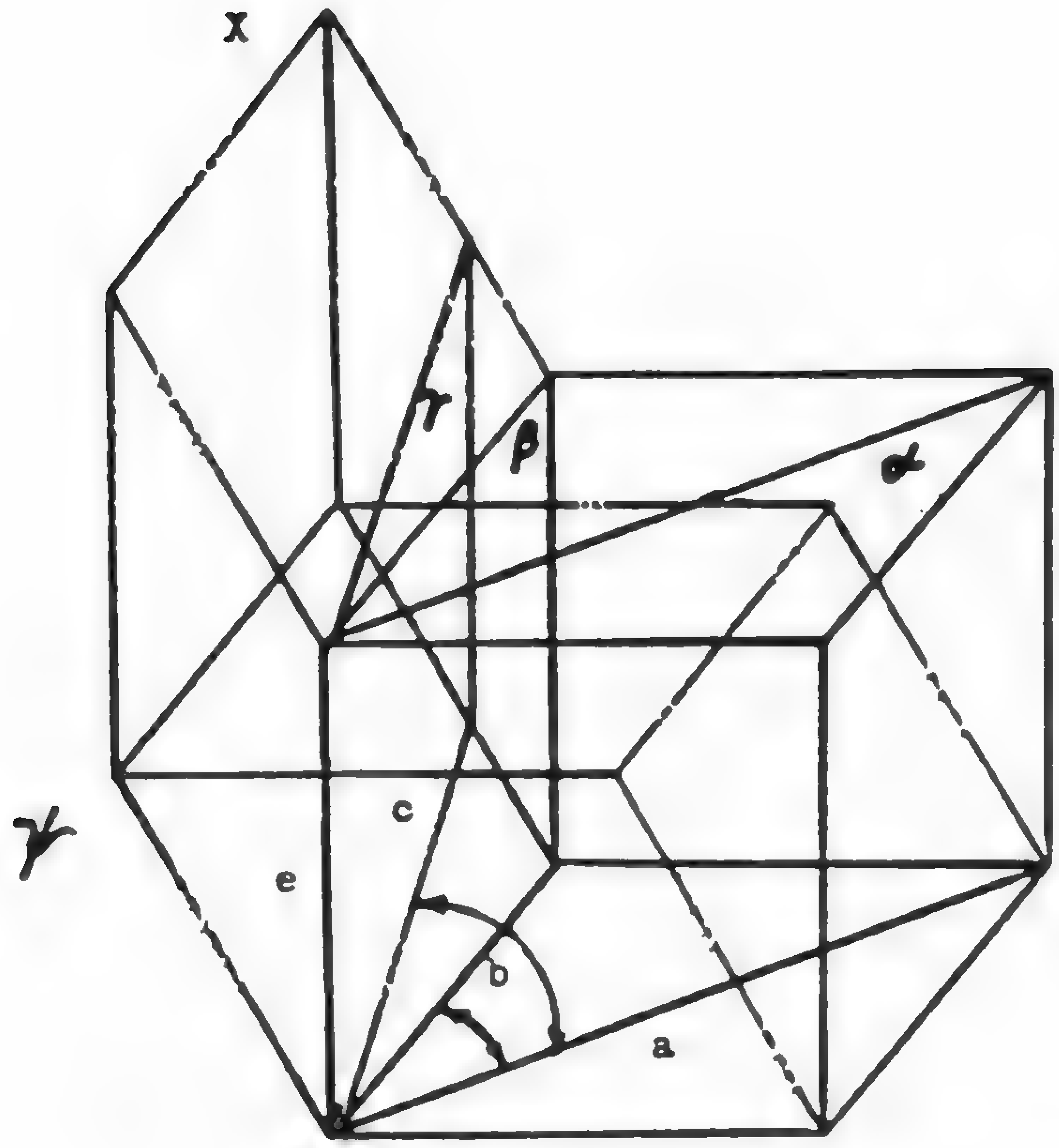
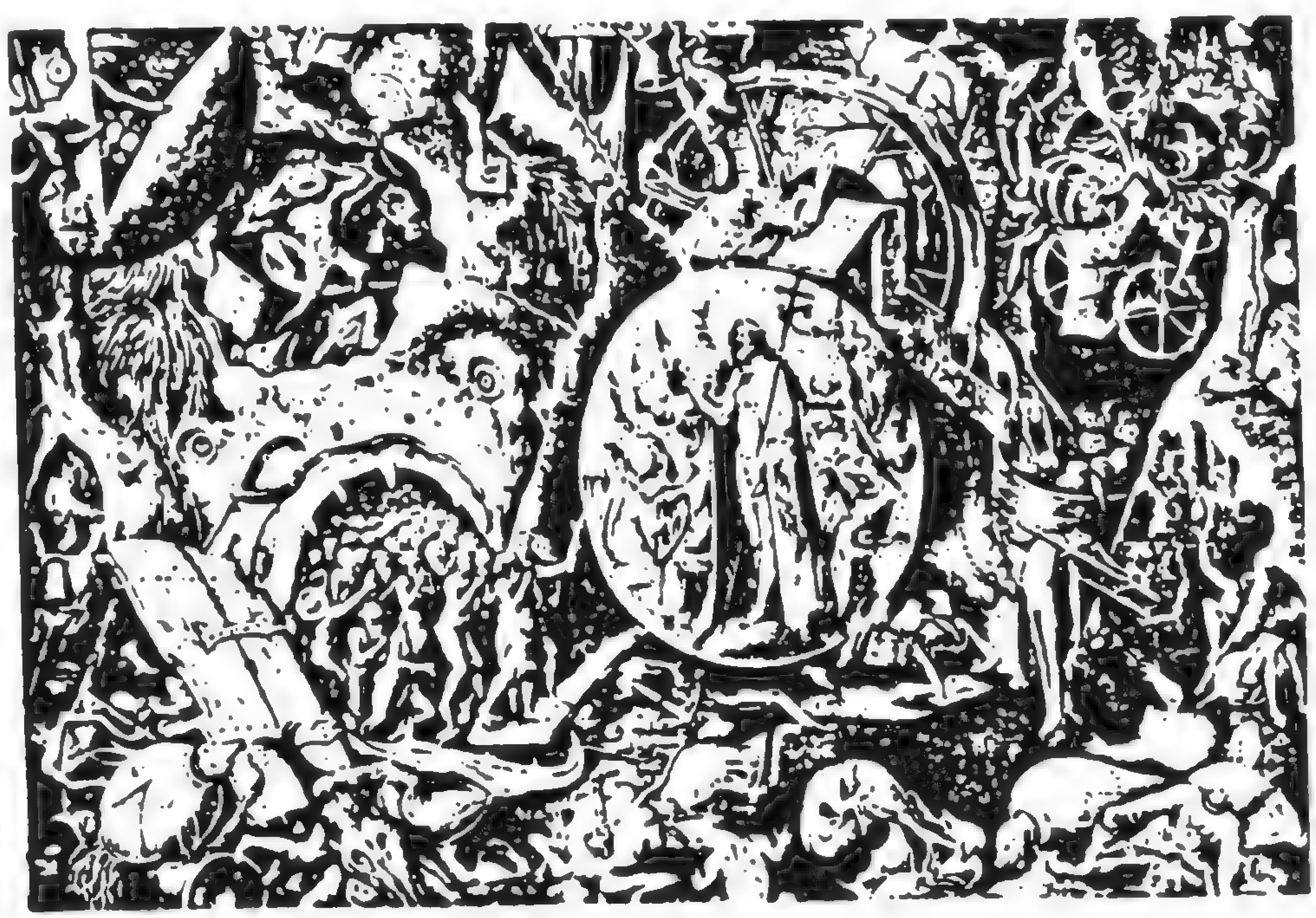


Fig. 48.

the projection of a upon X; and makes an angle with a which is less than the angle which it makes with c (Art. 14, Th. 2). Therefore the dihedral-angle which α makes with β is less than the dihedral-angle which it makes with γ. (Q.E.D)

When a ½-plane with its edge in a given hyperplane doe not lie in the hyperplane and is not perpendicular to it, the plane-angle of the dihedral-angle which it makes with the ½-plane having the same edge and containing its projection is called THE ANGLE OF THE ½-PLANE AND HYPERPLANE. When a ½-plane with its edge in a given hyperplane is perpendicular to the hyperplane, it is said to make a RIGHT-ANGLE WITH THE HYPERPLANE.



CHAPTER IV

ANGLES OF 2 PLANES AND ANGLES OF HIGHER ORDER

I. COMMON-PERPENDICULARS

34. THE COMMON-PERPENDICULAR LINE OF A LINE AND PLANE AND THE COMMON-PERPENDICULAR PLANE OF 2 PLANES WHICH HAVE A COMMON-PERPENDICULAR HYPERPLANE.

Theorem 1. Given a line a and a plane β not in the same hyperplane, there is a point of a whose distance from β is less than or equal to the distance from β of any other point of a, and the line along which we measure this minimum-distance is perpendicular to both. (Fig. 49.)

Given: A line a and a plane β not in the same hyperplane.

To Prove: There is a point of a whose distance from β is less than or equal to the distance from β of any other point of a, and the line along which we measure this minimum-distance is $\perp$ to both.

Proof: Let b be the line along which lies the projection of a upon β. Let c be the line $\perp$ to a and b at points P and Q respectively, then c will be $\perp$ to β at Q. Now a and b lie in a hyperplane (Art. 19, Th. 1), and the distance of any point of a from its projection upon β is less than its distance from any other point of β. Therefore, any point of a whose distance from b is less than or equal to the distance from b of any other point of a, will be a point whose distance from β is less than or equal to the distance from β of any other point of a; and the line along which we measure this minimum-distance will be $\perp$ to a and β, being $\perp$ to a and b. But c is this line; for, any line $\perp$ to a and β will be $\perp$ to b, and will be the projecting-line of a point of a. Therefore c being $\perp$ to a and β at P and Q respectively, is the line along which we measure this minimum-distance and is PQ. (Q.E.D)

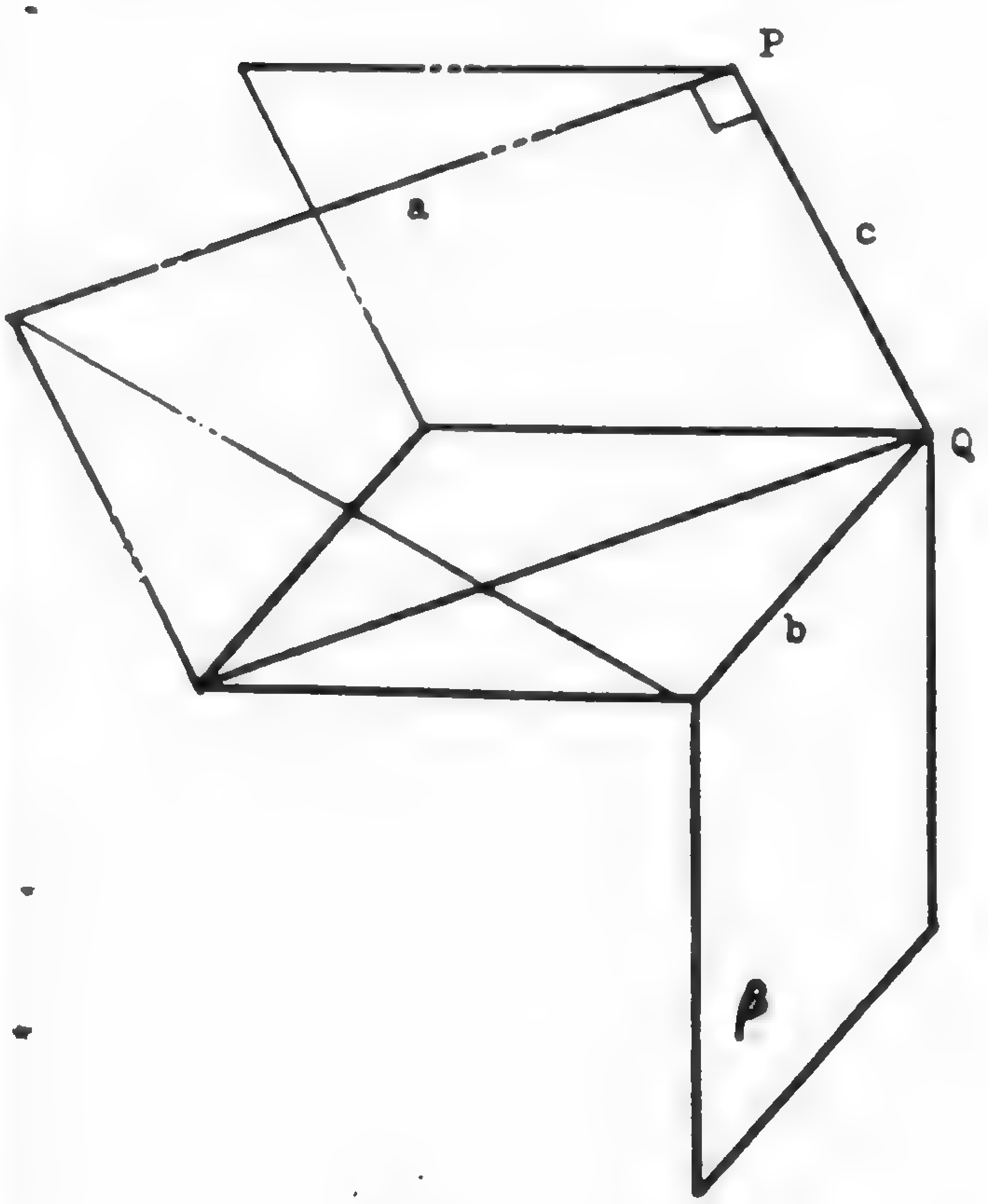


Fig. 49.

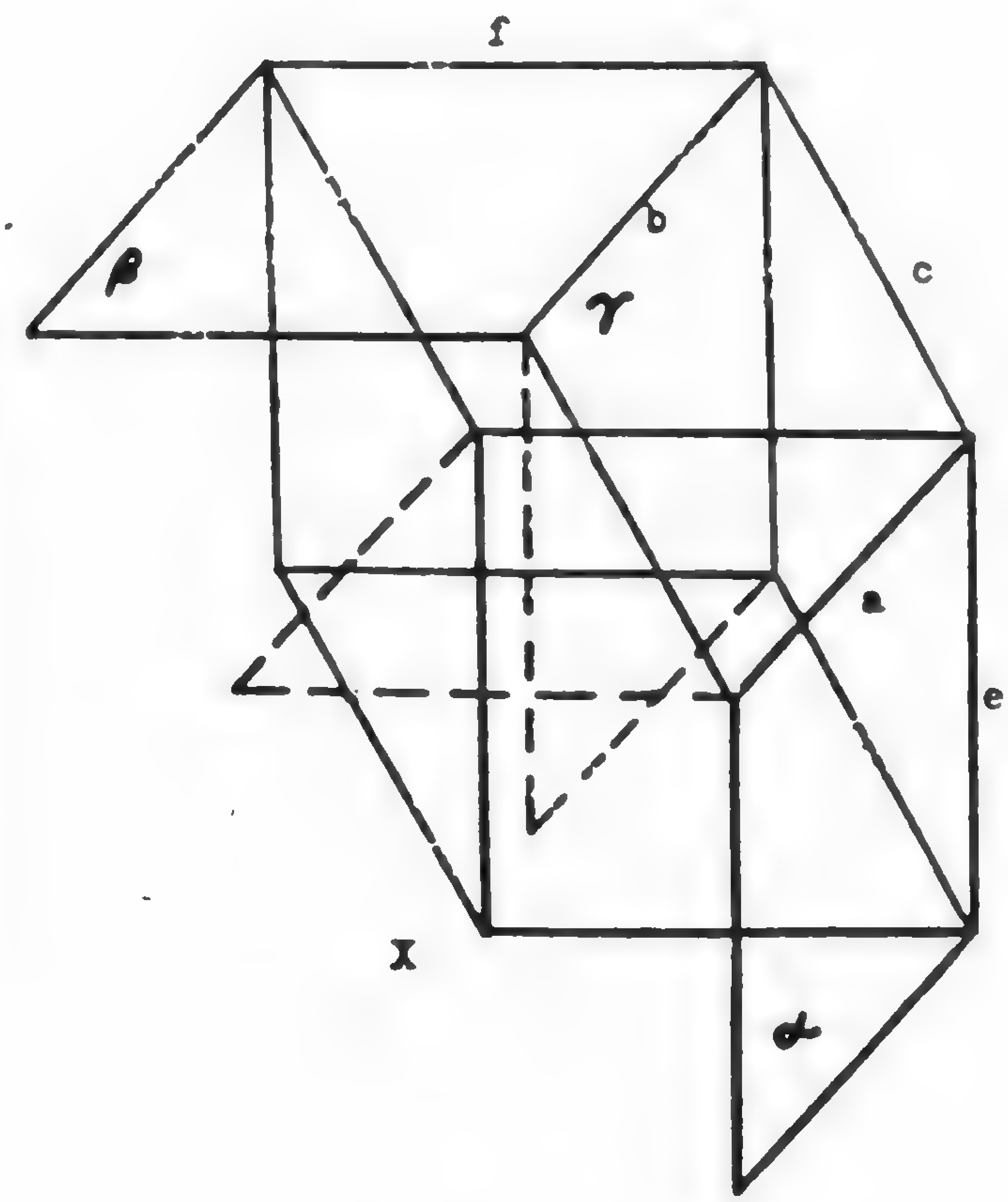


Fig. 50.

Theorem 2. If 2 planes not in 1 hyperplane have a common-perpendicular hyperplane (Art. 26), they have a common-perpendicular line and a common-perpendicular plane. (Fig. 50.)

Given: 2 planes α and β not in 1 hyperplane, with the hyperplane X being the common $\perp$ hyperplane to α and β.

To Prove: α and β have a common $\perp$ line and a common $\perp$ plane.

Proof: Let e and f be the lines of α and β respectively, in which the common $\perp$ hyperplane X intersects α and β. The lines e and f do not lie in 1 plane (Art. 4, Th. 3). e and f have a common $\perp$ line c. The line c is $\perp$ to α and β (Art. 24, Th. 2). Let γ be the plane containing c and intersecting the linear-elements of α and β in the elements a and b respectively, then γ will be the plane $\perp$ to α and β. Therefore α and β have a common $\perp$ line and a common $\perp$ plane. (Q.E.D)

We shall give a group of theorems related to the Elliptic-Geometry of Hyperspace as follows:

Theorem 1. If 2 lines not in the same plane have more than 1 common-perpendicular, they have an infinite-number of common-perpendiculars; along all of these perpendiculars the distance between them is the same, and any 2 of the perpendiculars cut-off the same distance on them. (Proof of this theorem can be found in Manning's Geometry of Four Dimensions in Art. 62, p-108; we shall develop the graphics of the hypersphere in another Chap. which will represent the Double-Elliptic Geometry of Hyperspace.).

Theorem 2. If a line and plane not in 1 hyperplane have more than 1 common-perpendicular line they have an infinite-number of these perpendiculars, 1 through every point of the line; along all of these perpendiculars the distance between them is the same, and any 2 of the perpendiculars intersect the line and plane at the same distance from each other.

Theorem 3. If 2 planes not in 1 hyperplane have a common-perpendicular and more than 1 common-perpendicular plane, they have an infinite-number of common-perpendicular planes, 1 through each linear-element; and any 2 of these planes are equidistant along the intersection of the given planes with the given perpendicular-hyperplane or with all the perpendicular-hyperplanes if there can be more than 1.

II POINT-GEOMETRY

35. A GEOMETRY WHOSE ELEMENTS ARE THE ½-LINES DRAWN FROM A GIVEN POINT. We shall make a study of the angles formed at a point O by the lines, planes, and hyperplanes which pass through O; and to simplify the presentation of the subject, we shall omit all mention of this point, it being understood that the lines, planes, and hyperplanes which pass through O are the only ones considered. Since all lines, planes, and hyperplanes have O in common, we can always say that 2 planes in a hyperplane intersect, and that any plane and hyperplane or any 2 hyperplanes intersect; it being understood that in each particular case the intersection exist.

We shall call the geometry of the various kinds of angles which we may have at a point POINT-GEOMETRY.

As O completely separates the 2 opposite ½-lines drawn from it along any line, we shall consider the ½-line as one of the elements of the Point-Geometry, rather than the entire-line. To every ½-line there is, then, 1 opposite; and a plane or hyperplane containing 1 of 2 opposite ½-lines contains the other. 2 planes in a hyperplane, or any plane and hyperplane, intersect in 2 opposite ½-lines. Each hyperplane has 2 perpendicular ½-lines, one opposite the other.

Point-Geometry in space of 4-dimensions is a 3-dimensional geometry. That is, we get all the elements of the Point-Geometry if we take 4 elements not in 1 hyperplane, all elements coplanar with any 2 of them, and all elements coplanar with any 2 obtained by this process. We can interpret the 2-dimensional and 3-dimensional geometries given in the plane- and solid-geometries as Point-Geometries, if we give a proper meaning to their undefined-terms and confine ourselves when necessary to a restricted angular-region.

III THE ANGLES OF 2 PLANES

36. TEOREMS OF PERPENDICULAR PLANES STATED IN THE LANGUAGE OF POINT-GEOMETRY. In this section we use the language of Point-Geometry, all planes and hyperplanes being assumed to pass through a given point O, and all $\frac{1}{2}$-lines to be drawn from O.

For perpendicular-planes we have proved certain theorems which can be stated here as follows (Art. 16, 17, 20, and 22):

Theorem 1. Each plane has 1 and only 1 absolutely-perpendicular plane.

Theorem 2. If 2 planes intersect, their absolutely-perpendicular planes intersect.

Theorem 3. A plane perpendicular to 1 of 2 absolutely-perpendicular planes is perpendicular to the other. (This theorem is included in Th. 4 that follows.)

Theorem 4. Given 2 pairs of absolutely-perpendicular planes, if either plane of 1 pair is perpendicular to 1 plane of the other pair, or if either plane of 1 pair intersects both planes of the other pair, then each plane of either pair intersects both planes of the other pair and is perpendicular to both planes of the other pair.

Theorem 5. If 2 planes have a common-perpendicular plane, the plane absolutely-perpendicular to the latter is also a common-perpendicular plane to the 2 given planes.

Theorem 6. 2 planes which intersect have 1 snd only 1 pair of common-perpendicular planes.

37. EXISTENCE OF COMMON-PERPENDICULAR PLANES.

Theorem. When 2 planes α and β do not intersect, the plane of the minimum-angle which a $\frac{1}{2}$-line of α makes with β is perpendicular to α and β.

It will be better to postpone the proof of this theorem until we have taken up a study of the Isocline-Planes of the Point-Geometry given in Chap. V (Art. 50).

We shall give a theorem of planes with an infinite-number of common-perpendicular planes.

Theorem. If 2 planes α and β cut-out equal angles on a pair of common-perpendicular planes, they have an infinite-number of common-perpendicular planes, the plane projecting any $\frac{1}{2}$-line of either upon the other being perpendicular to both. On all of these common-perpendicular planes they cut-out equal angles; and if α and β are not absolutely-perpendicular, any 2 of these planes cut-out on β angles equal to the angles which they cut-out on α.

Conversely, if α and β, not being absolutely-perpendicular, have more than 2 common-perpendicular planes, the acute-angles which they cut-out on any pair of these common-perpendicular planes are equal. (For proof of this theorem see Manning's Geometry of Four Dimensions, Art. 69, p-119.)

38. THE ANGLES BETWEEN 2 PLANES. ISOCLINE-PLANES. The acute-angles (or right-angles) ϕ and ϕ' which 2 planes cut-out on their common-perpendicular planes are called the angles between the 2 planes. When 1 of these angles is 0 the 2 planes intersect and lie in a hyperplane. The other angle is then the measure of the acute-dihedral-angles (or right-dihedral-angles) whose faces lie in the 2 planes. When 1 angle is 0 and 1 a right-angle the planes are simply-perpendicular. When 1 angle is a right-angle the planes are sometimes said to be perpendicular even if the other angle is not 0, but we shall use the word 'perpendicular' alone as applied to planes only when the other angle is 0. When both angles are right-angles the planes are absolutely-perpendicular.

In a study of the Point-Geometry (theorems and proofs) we shall make use of a LOGIC-DIAGRAM drawn to represent points at a given distance from O. $\frac{1}{2}$-lines (drawn from O) are represented in the diagram by points, and planes by lines. Angles in the diagram are represented by lengths of line-segments.

We shall use the logic-diagram (circuit) in 2 senses when it is associated with a graphic-form of the Point-Geometry: 1st, simply as $\frac{1}{2}$-lines (drawn from O) of a rectangular-system, and the planes determined by these $\frac{1}{2}$-lines as well as the angles at O; 2nd, as points of the hypersphere itself, that is, any 2 $\frac{1}{2}$-lines (drawn from O) will determine a portion of a plane that intercepts the hypersphere in an arc of a great-circle, and which is represented in the diagram by the length of a line-segment. However,

it must be understood that when we use the diagram in the 1st-sense, the translation from the graphic-form back to the diagram implies that we consider only the points on the $\frac{1}{2}$-lines at a given distance from O. The graphic and diagram relationships are in 1-to-1 correspondence. The 'concepts' of the Point-Geometry will become crystal-clear when we take up a study of the hypersphere.

In diagram 1, let a, b, c, and d be 4 mutually-perpendicular $\frac{1}{2}$-lines of a rectangular-

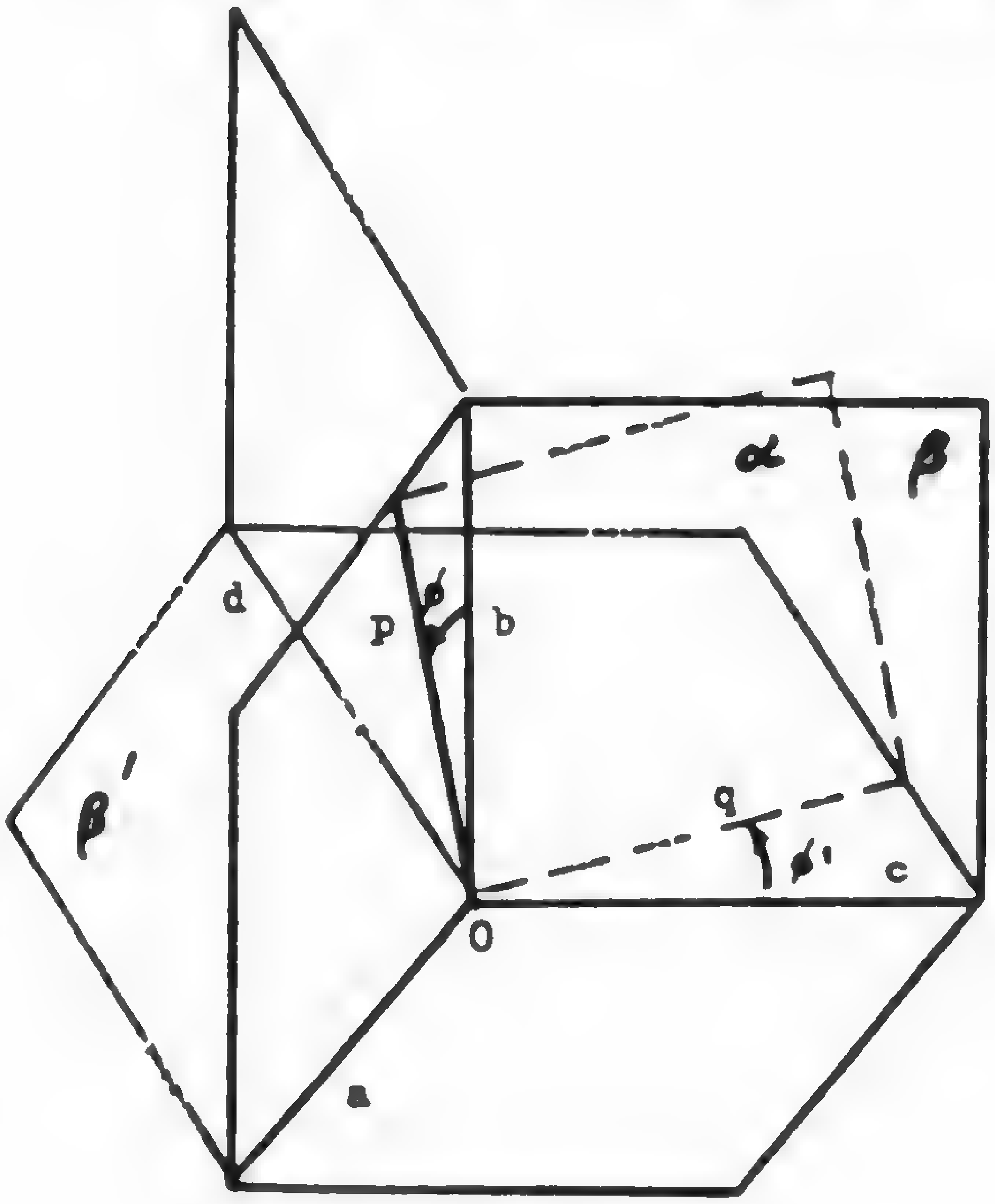
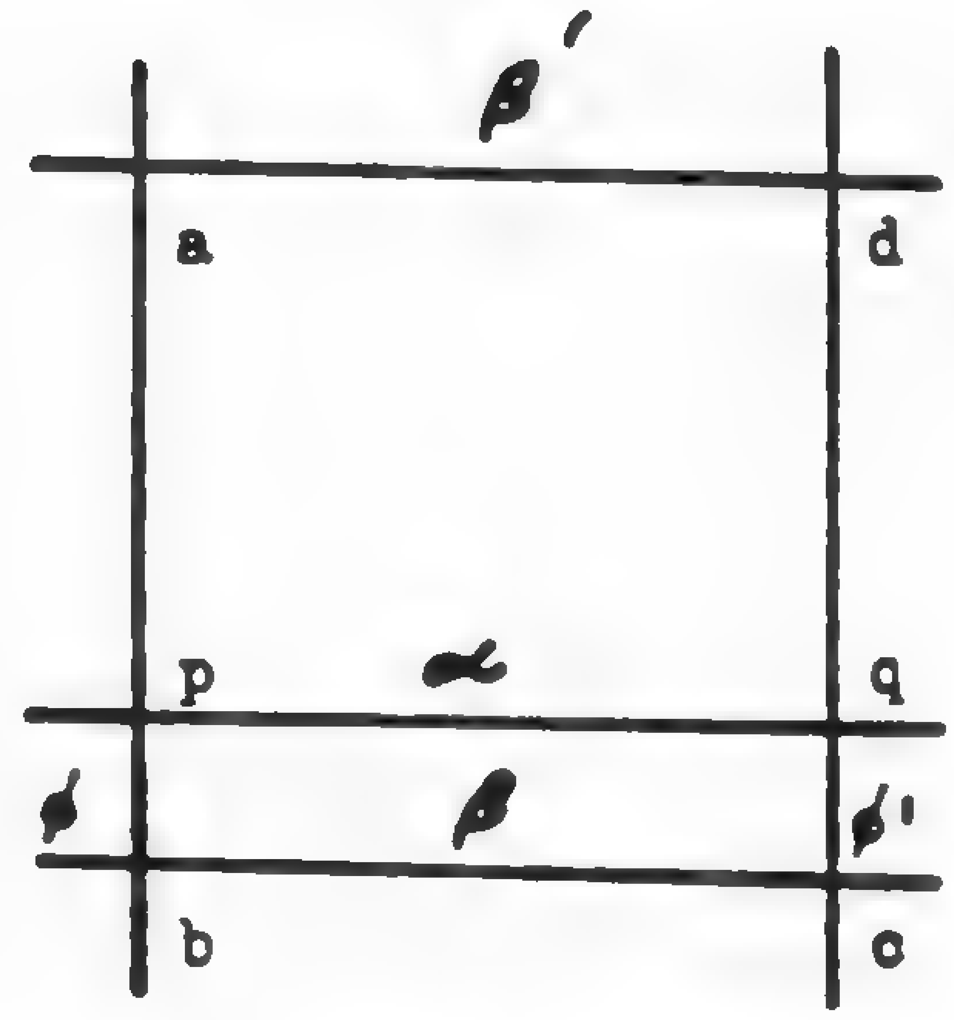


Fig. 52.



Logic-Diag. 1.

system. Suppose in the planes ba and cd we lay-off from b and c the angles ϕ and ϕ'. The $\frac{1}{2}$-lines which terminate these angles determine a plane α which makes with the plane of bc the angles ϕ and ϕ', ba and cd being the pair of common-perpendicular planes perpendicular to α and to bc.

Given any 2 planes α and β with their common-perpendicular planes γ and γ', we can take for bc the plane β and for ba and cd the planes γ and γ'. ad will be the plane β' absolutely-perpendicular to β, and the angles ϕ and ϕ' will be laid-off as above in the planes ba and cd.

When we say that the plane α makes with bc the angles ϕ and ϕ', we imply a sense of rotation in α corresponding to the order bc. If p and q are the terminal $\frac{1}{2}$-lines of these angles, then α is the plane pq, with a sense of rotation which turns p through 90° to the position of q.

The angles α makes with the plane ad are the complements of ϕ and ϕ'. We shall assume that a counter-clockwise-rotation (positive-direction) of the angles ϕ and ϕ' occurs when the $\frac{1}{2}$-lines (initial-side of the angles ϕ and ϕ') at b and c moves in the direction towards the $\frac{1}{2}$-lines a and d, that is, the plane β in the diagram slides towards the position of its absolutely-perpendicular plane at β'; in the graphic, then, the $\frac{1}{2}$-lines b and c rotate in a counter-clockwise-direction about O in the planes ba and cd respectively, and a 90° counter-clockwise rotation of b and c in planes ba and cd respectively, will take β to its position at β'.

In Fig. 52, the acute-angles ϕ and ϕ' which 2 planes α and β cut-out on their common-perpendicular planes ba and cd are called the angles between the 2 planes α and β.

When $\phi \neq 0$ and $\phi' = 0$ the 2 planes α and β intersect in the line c and lie in the hyperplane O-abc, with the angle ϕ then being the measure of the acute-dihedral-angle whose faces lie in the 2 planes α and β; like-results occur when $\phi = 0$ and $\phi' \neq 0$, but

with α and β in the hyperplane O-bcd.

When $\phi = 90^\circ$ and $\phi' = 0$ the planes α and β are then simply-perpendicular and lie in the hyperplane O-abc; like-results occur when $\phi = 0$ and $\phi' = 90^\circ$, in this case α and β lie in the hyperplane O-bcd.

When $\phi = \phi' = 90^\circ$ the planes α and β are then absolutely-perpendicular, that is, α takes the position of β' (coincides with β').

When $\phi = \phi'$, α is said to be ISOCLINE to the planes bc and ad; and by giving different values to ϕ we have an infinite-number of planes isocline to bc and ad and isocline to one another. Absolutely-perpendicular planes are always isocline to one another.

The hyperplanes O-abc and O-acd intersect in the plane O-ac. In the hyperplane O-abc, a counter-clockwise-rotation of ϕ in ba, with initial-side at b, moves in the direction around the point O in the plane bc towards a; and in the hyperplane O-acd, a counter-clockwise-rotation of ϕ' in cd, with initial-side at c, moves in the direction around the point O in the plane cd towards d.

The student should note the resemblance of the Point-Geometry to Vector-Mechanics, that is, for 'rotations'. In Vector-Mechanics, then, the rectangular-systems in the hyperplanes O-abc and O-acd are RIGHT-HANDED; such a system derives its name from the fact that, in the hyperplane O-abc, a right-threaded-screw rotated through 90° from O-b to O-a will advance in the positive c-direction, and in the hyperplane O-acd, a right-threaded-screw rotated through 90° from O-c to O-d will advance in the positive a-direction.

Theorem 1. The angles which a plane makes with 1 of 2 absolutely-perpendicular planes are the complements of the angles which it makes with the other; and any 2 planes make the same angles as their absolutely-perpendicular planes.

Proof of this theorem given in Chapter V at the end of section III on Isocline-Planes.

Theorem 2. If 2 $\frac{1}{2}$-lines in a plane α make equal angles with a plane β, the $\frac{1}{2}$-line bisecting the angle between them and the $\frac{1}{2}$-line bisecting the angle between their projections upon β will lie in 1 of the planes perpendicular to α and β.

Corollary. If more than 2 pairs of opposite $\frac{1}{2}$-lines in 1 of 2 planes makes any given angle with the other plane, the 2 planes are isocline.

IV. POLYHEDROIDAL-ANGLES

39. POLYHEDROIDAL-ANGLES. INTERIOR OF A POLYHEDROIDAL-ANGLE. A POLYHEDROIDAL-ANGLE consists of the $\frac{1}{2}$-lines drawn through the points of a polyhedron from a given point not in the hyperplane of the polyhedron, together with this given point. The $\frac{1}{2}$-lines are called ELEMENTS, the polyhedron is the DIRECTING-POLYHEDRON, and the point is the VERTEX. The elements which pass through the vertices of the polyhedron are called EDGES, the elements which pass through the points of any edge of the polyhedron lie in the interior of a FACE-ANGLE, and the elements which pass through the points of any face of the polyhedron lie within a POLYHEDRAL-ANGLE of the polyhedroidal-angle. The interiors of the polyhedral-angles are the CELLS. Adjacent polyhedral-angles are joined by their face-angles, these lying in the planes of intersection of their hyperplanes. A polyhedroidal-angle is CONVEX when each of these hyperplanes contains no element of the polyhedroidal-angle except those which belong to the polyhedral-angle of this hyperplane and to its interior. The polyhedroidal-angle is convex when the directing-polyhedron is convex. Only convex-polyhedroidal-angles will be considered in this treatise.

The INTERIOR OF A POLYHEDROIDAL-ANGLE consists of the $\frac{1}{2}$-lines drawn from the vertex through the points of the interior of the directing-polyhedron. The interior of a convex-polyhedroidal-angle lies within any 1 of its hyperplane-angles; and a point lying within all of the hyperplane-angles lies in the interior of the polyhedroidal-angle. The polyhedroidal-angle separates the rest of hyperspace into 2 portions, interior and exterior to it.

The polyhedroidal-angle whose elements are $\frac{1}{2}$-lines opposite to the elements of a given polyhedroidal-angle is VERTICAL to the latter. In 2 vertical-polyhedroidal-angles the face-angles, dihedral-angles, and polyhedral-angles of one are all vertical to the corresponding parts of the other, and the face-angles and dihedral-angles of one are

equal to the corresponding face-angles and dihedral-angles of the other.
A polyhedroidal-angle taken together with its vertical-polyhedroidal-angle may be reguarded as a particular-case of a hyperconical-hypersurface.

40. THETRAHEDROIDAL-ANGLES. THE RECTANGULAR-SYSTEM. The simplest polyhedroidal-angle is the TETRAHEDROIDAL-ANGLE, having a tetrahedron for directing-polyhedron. Any 4 hyperplanes with a point but not a line common to them all are the hyperplanes of a tetrahedroidal-angle, and any 4 ½-lines drawn from a point and not in 1 hyperplane are the edges of a tetrahedroidal-angle. The 4 hyperplanes, or the lines of the 4 ½-lines, determine a set of 16 tetrahedroidal-angles filling completely the hyperspace about the point, and associated in 8 pairs of vertical-tetrahedroidal-angles. In a tetrahedroidal-angle there are 6 planes, 3 pairs of opposites, the 2 planes of a pair meeting only at the vertex. Otherwise the 6 planes all intersect, 3 in each edge. We may also speak of each ½-line as opposite to the trihedral-angle formed by the other 3 ½-lines.

A point is in the interior of a tetrahedron if it is within any 3 of its dihedral-angles whose edges lie in 1 plane, or if it is within 2 dihedral-angles whose edges contain a pair of opposite-edges of the tetrahedron. In the same way, a point is in the interior of a tetrahedroidal-angle if it is within any 3 of its hyperplane-angles whose faces lie in 1 hyperplane, or if it is within 2 hyperplane-angles whose faces are the planes of 2 opposite-angles.

At each vertex of a pentahedroid is a tetrahedroidal-angle. A special-case is the rectangular-system: 4 ½-lines mutually-perpendicular, 6 face-angles which are right-angles lying in 3 pairs of absolutely-perpendicular planes, the rectangular-trihedral-angles, the right-dihedral-angles, and the 4 hyperplanes mutually-perpendicular.

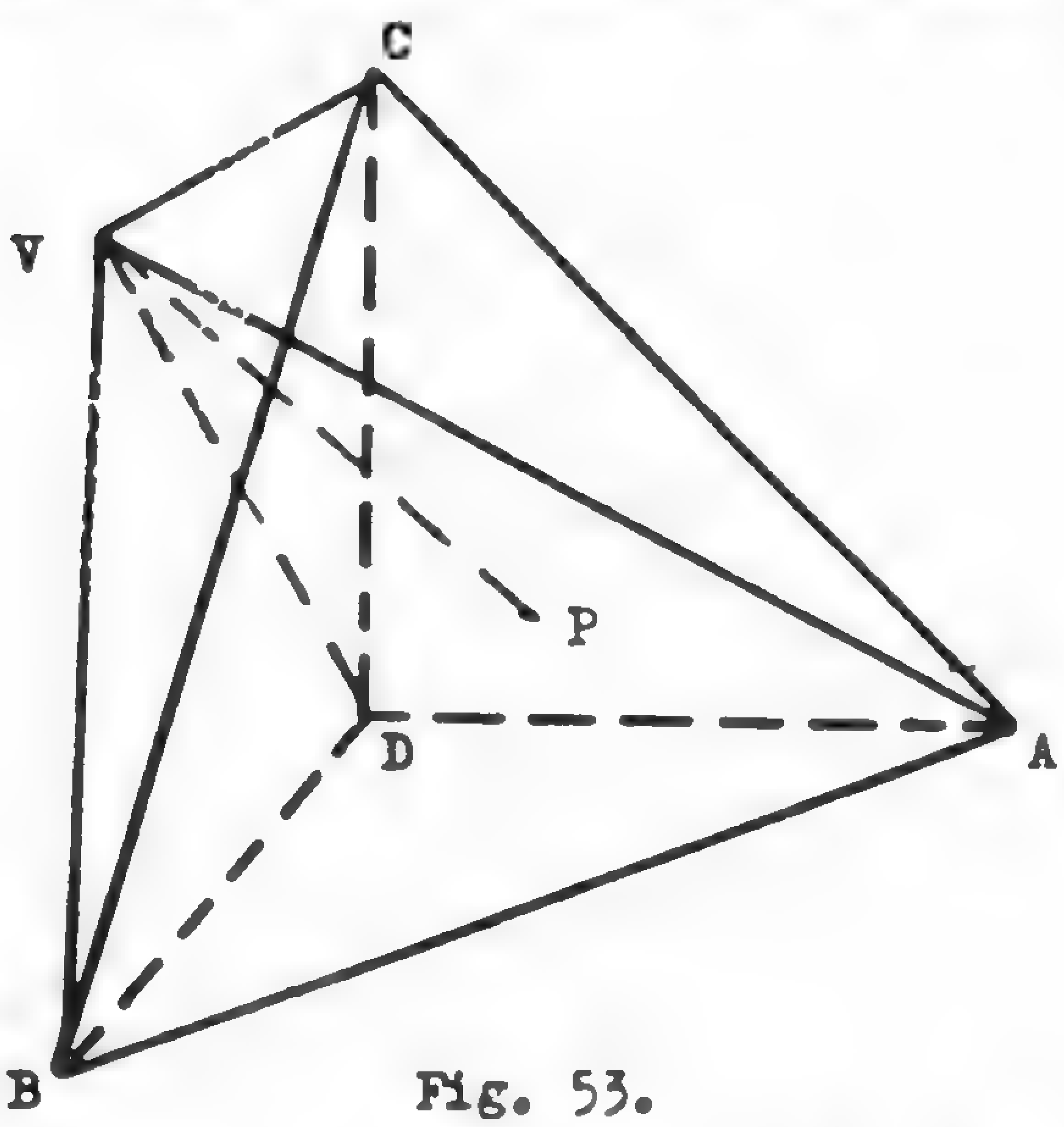


Fig. 53.

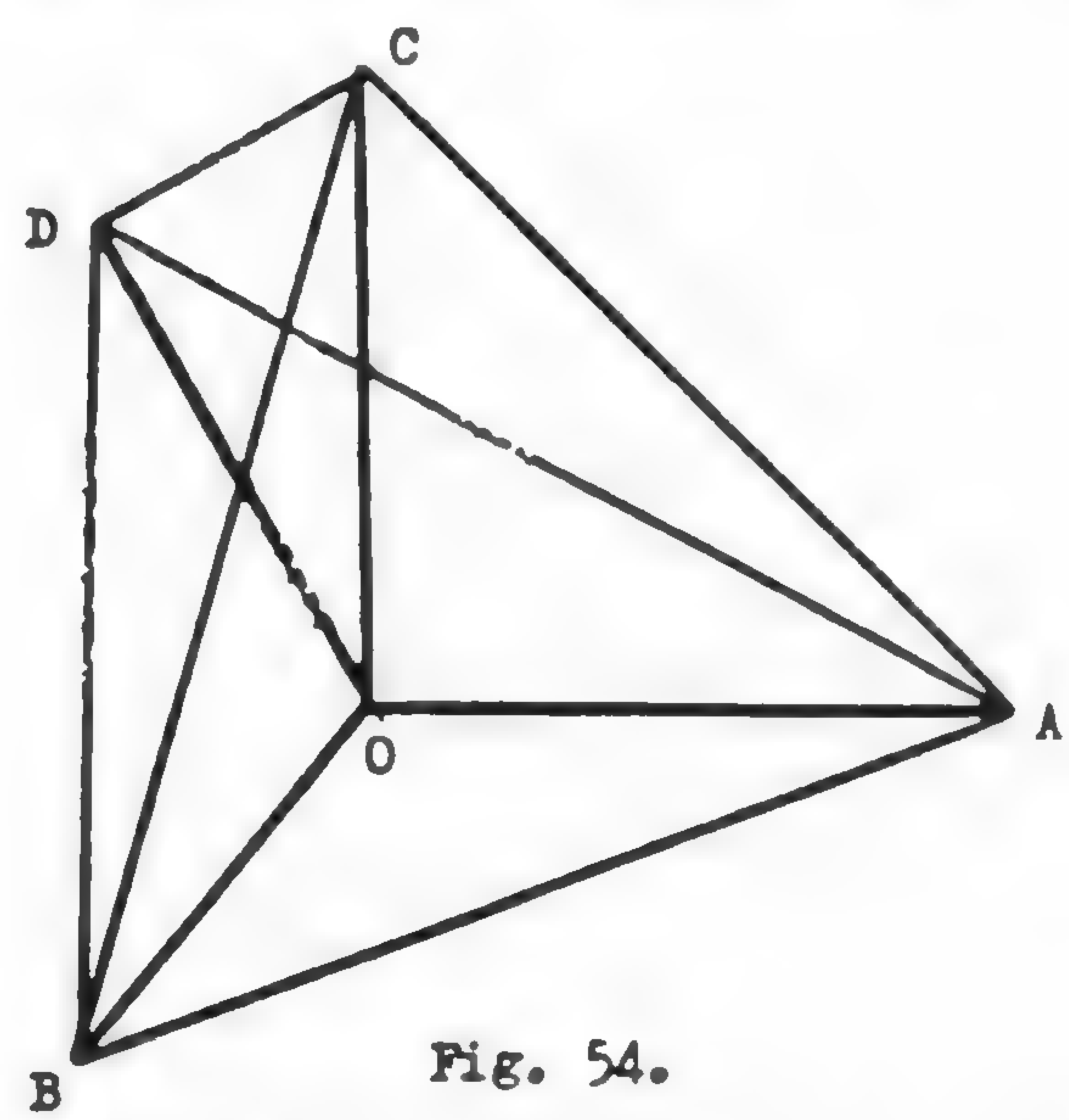


Fig. 54.

The tetrahedroidal-angle in Fig. 53 may be denoted by V-ABCD. The vertex is V, and the directing-tetrahedron is ABCD. The elements V-A(½-lines), V-B, V-C, and V-D are the edges of the tetrahedroidal-angle V-ABCD. The interiors of the trihedral-angles V-ABD, V-BCD, V-ACD, and V-ABC are the cells of V-ABCD.

Take any point P of the interior of the tetrahedron ABCD, then the element V-P lies in the interior of the tetrahedroidal-angle V-ABCD.

If we consider the rectangular-system at a point O, then the tetrahedroidal-angle in Fig. 54 may be denoted by O-ABCD. The 4 ½-lines O-A, O-B, O-C, and O-D mutually-perpendicular, 6 face-angles about the point O which are right-angles lying in 3 pairs of absolutely-perpendicular planes; the rectangular-trihedral-angles about O, the right-dihedral-angles about O, and the 4 hyperplanes about O mutually-perpendicular.

Other properties of the tetrahedroidal-angle can be immediately read-out from the graphic-forms of the tetrahedroidal-angles given in Figs. 53 or 54. For example, in Fig. 53, the planes VBD and VAC of V-ABCD are opposite to one another and which intersect only in the point V and are not absolutely-perpendicular to one another; whereas, in Fig. 54, the planes OAB and OCD of O-ABCD are opposite to one another and which intersect only in the point O and are absolutely-perpendicular to one another.

A hyperpyramid, or the hypersolid which we call the interior of a hyperpyramid, may be described as cut from the interior of a polyhedroidal-angle by a hyperplane that cuts all of its elements and does not pass through its vertex. (See Fig. 53, for the case when the polyhedroidal-angle is a tetrahedroidal-angle.)

41. THEOREMS ON TETRAHEDROIDAL-ANGLES.

Theorem 1. If each of the face-angles of a tetrahedroidal-angle is equal to the corresponding face-angles of a 2nd tetrahedroidal-angle, when the 4 edges of one are made to correspond in some order to the 4 edges of the other, then corresponding dihedral-angles and corresponding hyperplane-angles will all be equal.

Theorem 2. The $\frac{1}{2}$-hyperplanes bisecting the 6 hyperplane-angles of a tetrahedroidal-angle contain a $\frac{1}{2}$-line common to them all, the locus of points within the tetrahedroidal-angle equidistant from the 4 hyperplanes.

The proofs of these 2 theorems can be found in Manning's Geometry of Four Dimensions—Chap. III, Arts. 72 & 73.

V. PLANO-POLYHEDRAL-ANGLES

42. PLANO-POLYHEDRAL-ANGLES. PLANO-TRIHEDRAL-ANGLES. A PLANO-POLYHEDRAL-ANGLE consists of a finite-number of $\frac{1}{2}$-planes with a common-edge in which these $\frac{1}{2}$-planes are in a definite cyclical-order, together with their common-edge and the interiors of the dihedral-angles whose faces are consecutive $\frac{1}{2}$-planes of this order. The $\frac{1}{2}$-planes are the FACES, their common-edge is the VERTEX-EDGE, and the interiors of the dihedral-angles are the CELLS. If α, β, γ, ... are the faces in order, the cells are the interiors of the dihedral-angles $\alpha\beta$, $\beta\gamma$, ..., and the plano-polyhedral-angle may be described as the plano-polyhedral-angle $\alpha\beta\gamma$... .

$\frac{1}{2}$-planes which lie within the cells and have the vertex-edge for edge, and those which are the faces of the plano-polyhedral-angle, are all called ELEMENTS and are in cyclical-order.

A plano-polyhedral-angle is SIMPLE when no $\frac{1}{2}$-plane occurs twice as an element—we shall alaways assume that it is simple. A simple plano-polyhedral-angle is CONVEX when the hyperplane of each cell contains no element except those of the cell itself and the 2 faces of the cell. Each face of a convex plano-polyhedral-angle is a $\frac{1}{2}$-plane lying in the common-face of 2 $\frac{1}{2}$-hyperplanes which contain 2 adjacent-cells. These 2 $\frac{1}{2}$-hyperplanes are the cells of a hyperplane-angle, 1 of the HYPERPLANE-ANGLES OF THE PLANO-POLYHEDRAL-ANGLE.

In a polyhedroid each edge lies in the vertex-edge of a plano-polyhedral-angle whose cells contain the adjacent-cells of the polyhedroid, and the edges of a polyhedroidal-angle lie in the vertex-edges of plano-polyhedral-angles which belong to the polyhedroidal-angle.

The plano-polyhedral-angle whose elements are $\frac{1}{2}$-planes opposite to the elements of a given plano-polyhedral-angle is VERTICAL to the latter. In 2 vertical plano-polyhedral-angles the dihedral-angles and hyperplane-angles of one are all vertical to the corresponding parts of the other.

A plano-polyhedral-angle with 3 faces is called a PLANO-TRIHEDRAL-ANGLE. Any 3 $\frac{1}{2}$-planes having a common-edge but not lying in 1 hyperplane are the faces of a plano-trihedral-angle. Any 3 hyperplanes which intersect but have only a line common to all 3 are the hyperplanes of a plano-trihedral-angle. The planes of 3 such $\frac{1}{2}$-planes, or 3 such hyperplanes, determine 8 plano-trihedral-angles completely filling the hyperspace about their line of intersection, and associated in 4 pairs of vertical plano-trihedral-angles.

The plano-polyhedral-angles of a pentahedroid are all plano-trihedral-angles.

In Fig. 54 the plano-trihedral-angle may be denoted by CD-ABO. The vertex-edge is CD. The consecutive faces are CD-A, CD-B, and CD-O. The cells are the interiors of the dihedral-angles A-CD-B, B-CD-O, and O-CD-A.

43. POLYHEDRAL-SECTIONS OF A PLANO-POLYHEDRAL-ANGLE. RIGHT-SECTIONS. A hyperplane intersecting the edge of a plano-polyhedral-angle, but not containing the edge, intersects the faces in $\frac{1}{2}$-lines which are the edges of a polyhedral-angle; and the

plano-polyhedral-angle may be considered as determined by a polyhedral-angle and a $\frac{1}{2}$-line through its vertex not in its hyperplane. When either the plano-polyhedral-angle or the polyhedral-angle is convex the other is convex.

A plano-polyhedral-angle might be regarded as a polyhedroidal-angle with a directing-polyhedral-angle instead of a directing-polyhedron: that is, the $\frac{1}{2}$-lines drawn through the points of a polyhedral-angle from a given point not in its hyperplane form a cetrain-portion of a plano-polyhedral-angle.

In Fig. 55, the hyperplane of the tetrahedron FGHO intersects the edge DE of the plano-trihedral-angle DE-ABC in the point O, and the faces DE-A, DE-B, and DE-C in the -lines O-F, O-G, and O-H respectively, and these $\frac{1}{2}$-lines which meet at O are the edges of a trihedral-angle O-FGH. We can consider the plano-trihedral-angle DO-FGH as determined by the trihedral-angle O-FGH and the $\frac{1}{2}$-line O-D through its vertex O not in its hyperplane.

Theorem. A hyperplane perpendicular at a point O to the vertex-edge of a plano-polyhedral-angle intersects the latter in a polyhedral-angle whose face-angles are the plane-angles at O of the dihedral-angles of the plano-polyhedral-angle, and whose dihedral-angles have at O the same plane-angles as the hyperplane-angles of the plano-polyhedral-angle. (Fig. 56.)

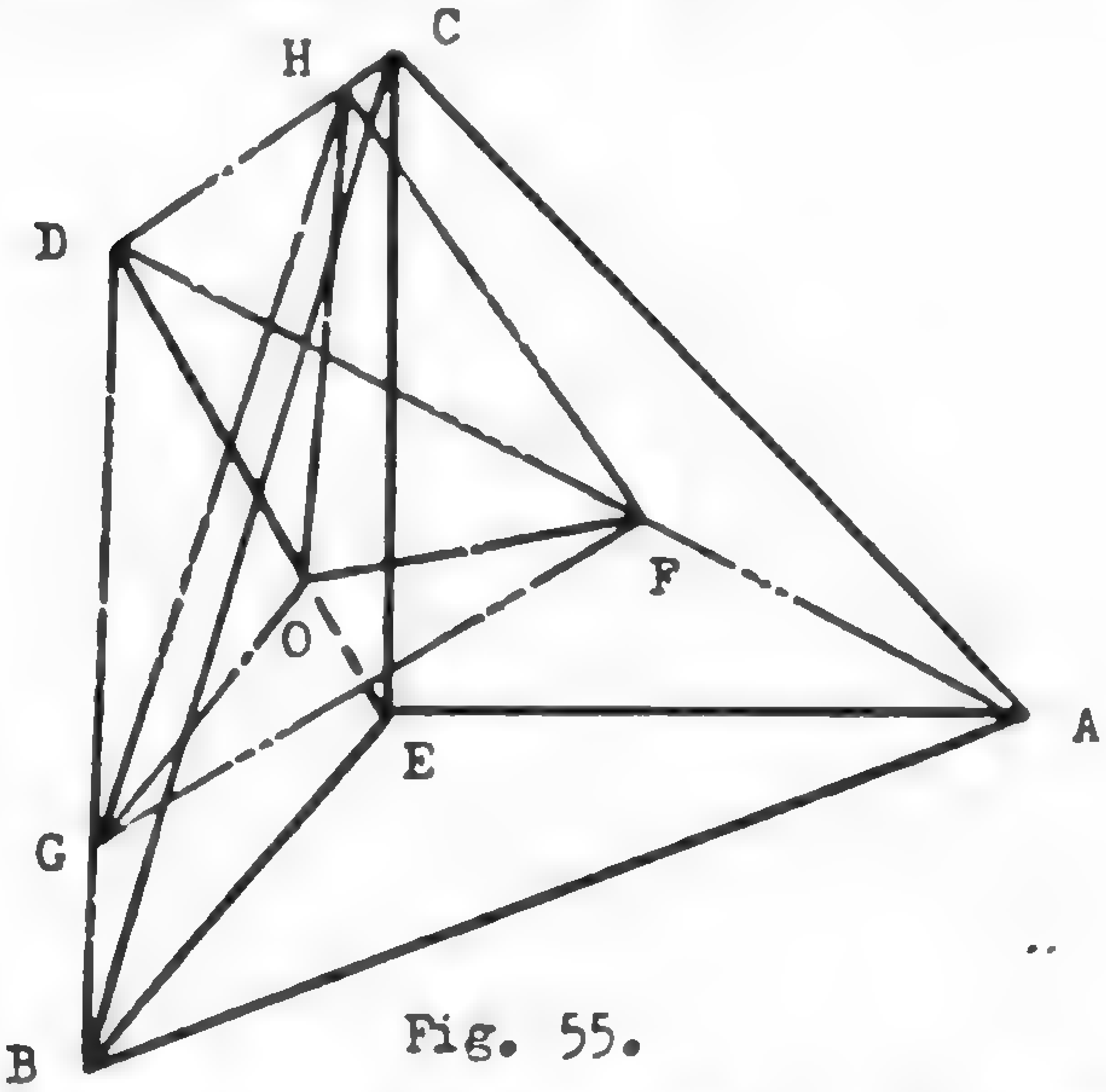


Fig. 55.

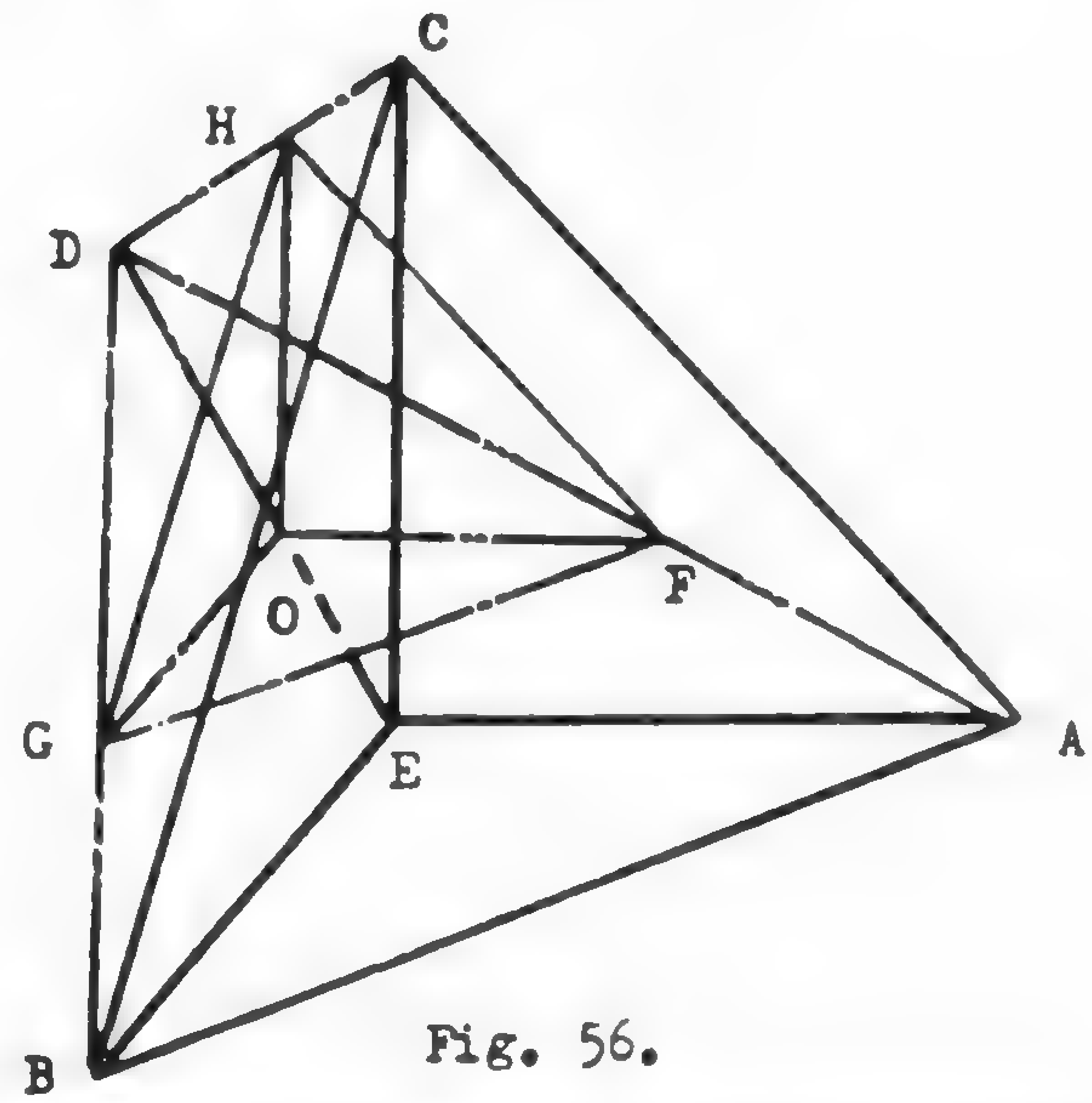


Fig. 56.

The proof of this theorem will be given for the special-case of a plano-trihedral-angle. A slight-modification in this proof will make it hold for a plano-polyhedral-angle of an arbitrary number n of faces.

Since the plano-polyhedral-angles of a pentahedroid are plano-trihedral-angles we shall make use of the graphic of a 'portion' of a pentahedroid in the proof of this theorem. In Figs. 55 and 56, the cells ABCE and ABCD are missing from the pentahedroid ABCDE.

Given: A hyperplane FGHO $\perp$ at a point O to the vertex-edge DE of a plano-trihedral-angle DE-ABC.

To Prove: FGHO intersects DE-ABC in a trihedral-angle O-FGH whose face-angles $\angle$ FOG, $\angle$ GOH, and $\angle$ HOF are the plane-angles at O of the dihedral-angles A-DE-B, B-DE-C, and C-DE-A, respectively, and whose dihedral-angles have at O the same plane-angles as the hyperplane-angles of DE-ABC.

Proof: The hyperplane FGHO, being $\perp$ to the vertex-edge DE at O, intersects the hyperplanes of the dihedral-angles A-DE-B, B-DE-C, and C-DE-A in the planes FOG, GOH, and HOF, respectively, and which are $\perp$ to the common-edge DE at O. Therefore the face-angles $\angle$ FOG, $\angle$ GOH, and $\angle$ HOF are the plane-angles of the dihedral-angles A-DE-B, B-DE-C, and C-DE-A, respectively, of DE-ABC. The hyperplane FGHO is then $\perp$ to the planes of the faces DE-A, DE-B, and DE-C of DE-ABC (Art. 25, Th. 1). The planes of the faces DE-A, DE-B, and DE-C are the faces of the hyperplane-angles B-ADE-C, A-BDE-C,

and A-CDE-B, respectively, of DE-ABC; therefore the dihedral-angles A-DE-B, B-DE-C, and C-DE-A of DE-ABC have at O the same plane-angles as the hyperplane-angles of DE-ABC (Art. 27, Th. 2). Therefore the theorem is proved. (Q.E.D)

The polyhedral-angle in which the plano-polyhedral-angle is cut by a hyperplane perpendicular to the edge is a RIGHT-SECTION of the plano-polyhedral-angle.

In Fig. 56, the trihedral-angle O-FGH is a right-section of the plano-polyhedral-angle DE-ABC.

44. THEOREMS ON PLANO-POLYHEDRAL- AND PLANO-TRIHEDRAL-ANGLES.

Theorem 1. The sum of 2 dihedral-angles of a plano-trihedral-angle is greater than the 3rd.

Theorem 2. The sum of the dihedral-angles of a convex plano-polyhedral-angle is less than the sum of 4 right-dihedral-angles.

Theorem 3. If 2 plano-trihedral-angles have the 3 dihedral-angles of one equal respectively to the 3 dihedral-angles of the other, their homologus hyperplane-angles are equal; and if 2 plano-trihedral-angles have 2 dihedral-angles and the included hyperplane-angle of one equal respectively to 2 dihedral-angles and the included hyperplane-angle of the other, the remaining parts of one are equal to the remanining parts of the other.

The above theorems can be proved by means of a right-section.

45. THE DIRECTING-POLYGON AND ½-PLANE ELEMENTS OF A PLANO-POLYHEDRAL-ANGLE. A convex plano-polyhedral-angle may be considered as determined by a convex-polygon and a line not in the hyperplane with the plane of the polygon. The elements of the plano-polyhedral-angle are then ½-planes having the given line for common-edge and containing each a point of the polygon. The polygon is called the DIRECTING-POLYGON, and each side of the polygon lies in the interior of a cell of the plano-polyhedral-angle.

2 vertical plano-polyhedral-angles taken together may be regarded as a special-case of a plano-conical-hypersurface.

In Fig. 53, we can consider the plano-trihedral-angle CV-ABD as determined by a triangle ABD and a line CV not in the hyperplane ABCD with the plane of the triangle ABD. The elements of CV-ABD are then ½-planes having the line CV for common-edge and containing each a point of ABD. The triangle ABD is the directing-triangle of the plano-trihedral-angle CV-ABD, and each side of the triangle ABD lies in the interior of a cell.

A double-pyramid, or the hyperpyramid which we call the interior of a double-pyramid may be described as cut from the interior of a plano-polyhedral-angle by 2 hyperplanes which contain the directing-polygon and each a point of the vertex-edge.

In Fig. 53, the double-pyramid CV-ABD is cut from the interior of a plano-trihedral-angle 'CV-ABD' by 2 hyperplanes CABD and VABD which contain the directing-triangle ABD, with CABD containing a point C of the vertex-edge CV and VABD containing a point V of the vertex-edge CV.

46. THE INTERIOR OF A CONVEX PLANO-POLYHEDRAL-ANGLE. A convex plano-polyhedral-angle divides the rest of hyperspace into 2 parts, INTERIOR and EXTERIOR. The interior contains the interiors of all segments whose points are points of the plano-polyhedral-angle except those whose interiors also lie in it. The interior belongs to the interior of each of its hyperplane-angles; and any point which is in the interior of each of the hyperplane-angles of a convex plano-polyhedral-angle is in the interior of the latter. In the case of a plano-trihedral-angle, if a point is in the interior of 2 of the hyperplane-angles it is in the interior of the plano-trihedral-angle.

Theorem. The 3 ½-hyperplanes which bisect the hyperplane-angles of a plano-trihedral-angle intersect in a ½-plane, the locus of points in the interior of the plano-trihedral-angle equidistant from the hyperplanes of its cells.

CHAPTER V

I. ROTATION

47. ROTATION IN HYPERSPACE. THE AXIS-PLANE. DOUBLE-ROTATION. In space of 4-dimensions 1 of 2 absolutely-perpendicular planes rotating on itself around the point where the 2 planes meet always remains absolutely-perpendicular to the other. We can say, then, that the rotating plane rotates around the other plane, and we can call the other plane an AXIS-PLANE. 2 planes absolutely-perpendicular to a given plane at points O and O' lie in a hyperplane in which they are perpendicular to the line OO' (Art. 18, Th.). Thus we can compare the rotations of 2 planes absolutely-perpendicular to the axis-plane by considering them as rotations in a hyperplane around an axis-plane.

In order to prove that any figure in hyperspace can rotate around a plane, we shall make use of a theorem from the 3-dimensional geometry.

Theorem. When all the planes perpendicular to a line in a hyperplane rotate around the line in the same-direction and through the same-angle or at the same-rate, figures in the hyperplane remain invariable, any 2 points being always at the same-distance from each other. (See Fig. 57, and use the hyperplane of the black-line-renderings.)

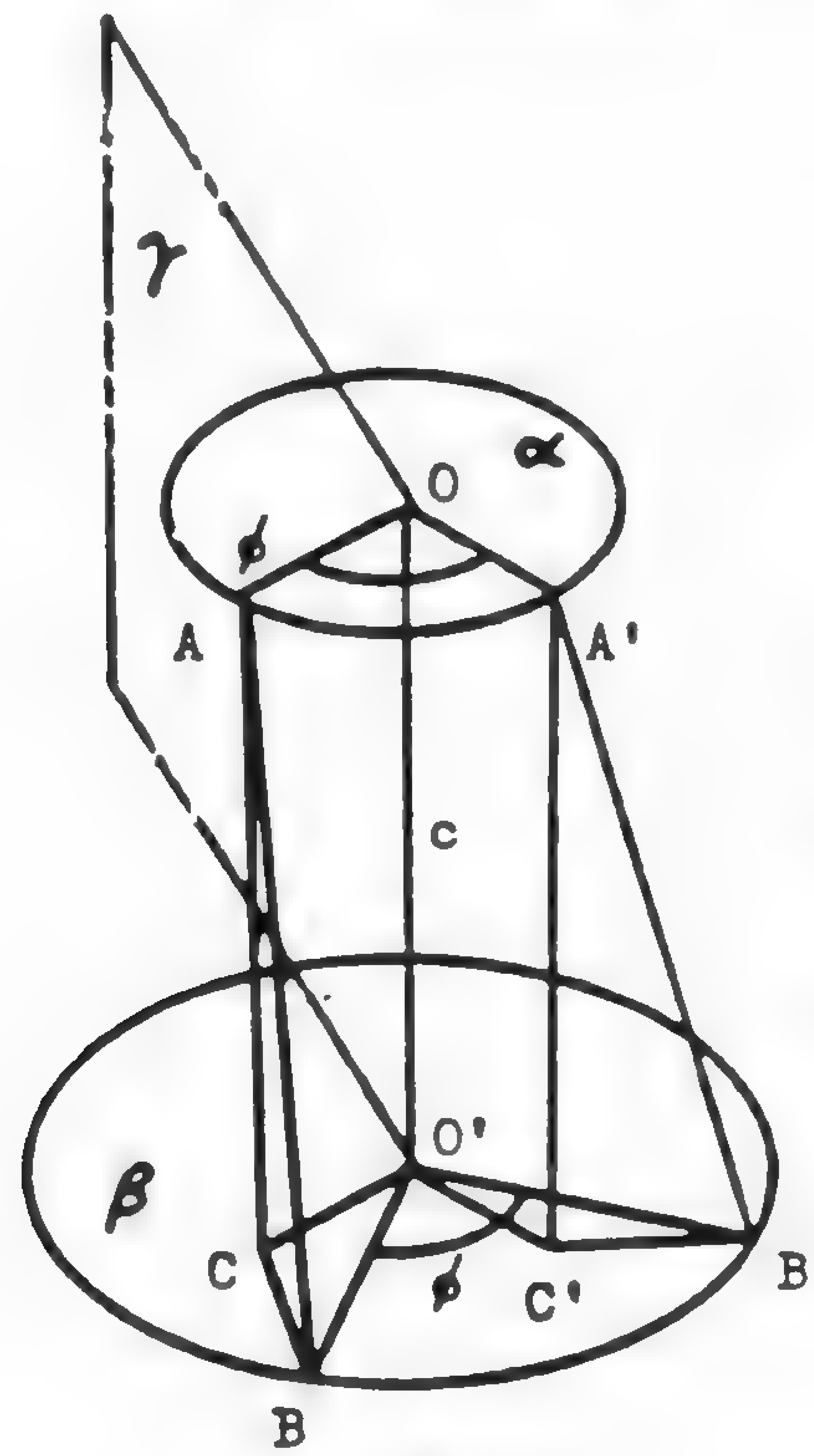


Fig. 57.

The proof of this theorem becomes equivalent to proving that any 2 points in a hyperplane have the same distance from each other after a rotation through a given angle.

Given: Any 2 distinct points A and B and their projections O and O' upon the axis-line c, and A' and B' their positions after a rotation through an angle ϕ.

To Prove: A'B' = AB.

Proof: The $\frac{1}{2}$-plane OO'-B rotates in the same-direction and through the same-dihedral-angle ϕ as the $\frac{1}{2}$-plane OO'-A. That is, A-OO'-A' = B-OO'-B', and therefore A-OO'-B = A'-OO'-B'.

Let α and β be the planes in which A and B rotate around the axis-line c, with α and β being $\perp$ to c, and let C and C' be the projections of A and A' upon β. Then the angles CO'B and C'O'B' are the plane-angles of A-OO'-B and A'-OO'-B' respectively, and angle CO'B = angle C'O'B'. Thus we prove that triangle CO'B = triangle C'O'B', and then that the right-angle ACB = the right-angle A'C'B'. Therefore A'B' = AB. (Q.E.D)

Thus we have proved that any figure in a hyperplane can rotate around a line, and we can think of the entire hyperplane as rotating on itself around 1 of its planes.

Theorem 1. When all the planes absolutely-perpendicular to a given plane rotate around the given plane as axis-plane in the same-direction and through the same-angle or at the same-rate, all figures remain invariable, any 2 points being always at the same-distance from each other. (Fig. 57.)

Given: Any 2 distinct points A and B, with the planes α and β $\veebar$ to a plane γ and passing through them, and O and O' their projections upon the axis-plane γ, and A' and B' their positions after a rotation through an angle ϕ.

Proof: The points A and B, with the planes α and β through them $\veebar$ to the plane γ at points O and O', lie in a hyperplane in which the rotation takes place around a line c $\perp$ to α and β (Art. 43, Th.). Therefore by the preceding theorem, the distance AB between the 2 points A and B remains unchanged. Therefore A'B' = AB. (Q.E.D)

Theorem 2. Rotations around 2 absolutely-perpendicular planes are commutative; after 2 such rotations all points of hyperspace take the same positions, whichever rotation comes first.

We shall call a combination of rotations around 2 absolutely-perpendicular planes a DOUBLE-ROTATION.

II. SYMMETRY

48. SYMMETRICAL-POSITIONS. SYMMETRY IN A HYPERPLANE. 2 points are SYMMETRICALLY-SITUATED WITH RESPECT TO THE POINT which lies midway between them. The point midway is called their CENTER OF SYMMETRY. 2 points are SYMMETRICALLY-SITUATED WITH RESPECT TO A LINE, PLANE, OR HYPERPLANE which is perpendicular to the line of the 2 points at their center of symmetry. Such a line, plane, or hyperplane is called the LINE, PLANE, OR HYPERPLANE OF SYMMETRY.

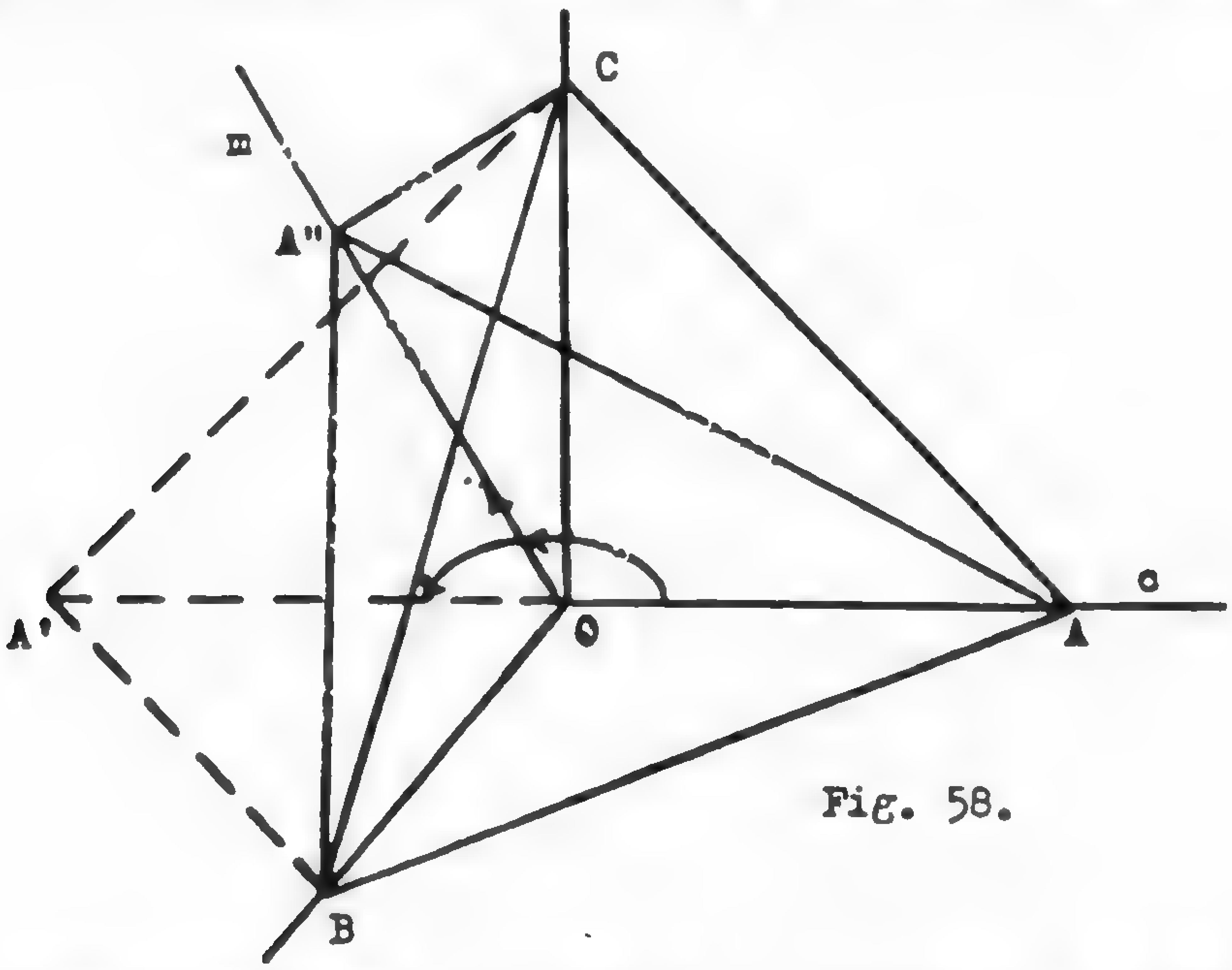


Fig. 58.

Theorem. When a hyperplane-figure is rotated in hyperspace through 180° around some plane of its hyperplane (Art. 47), it comes again into the same hyperplane to a position which is symmetrical to its 1st-position with respect to the plane; and so 2 figures in a hyperplane symmetrically-situated with respect to a plane of the hyperplane can be made to coincide by a rotation of 1 of them through 180° around the plane of symmetry. (Fig. 58.)—special-case.

In Fig. 58, let the trihedral-angle O-ABC at the vertex-point O of the tetrahedron OBCA be a rectangular-trihedral-angle, then the edge AO will be $\perp$ to the face OBC. Let m be the line $\perp$ to the hyperplane of the tetrahedron OBCA at O. Let α be the plane containing m and intersecting the hyperplane of the tetrahedron OBCA in the line c containing the edge AO of this tetrahedron, then a plane β $\perp$ to c at O will contain the face OBC and will be $\veebar$ to α at O (Art. 23, Th. 2). Now if we rotate the vertex-point A lying in α (the plane of mc) through 180° around the plane β (the plane of OBC) as axis-plane, the point A will again come into the hyperplane of the tetrahedron OBCA to a position opposite to the position of the point A; that is, A will occupy the position

of A', with A' and A symmetrically-situated with respect to the point O in the plane of symmetry β. Therefore, the tetrahedron OBCA rotated around its face OBC through 180° will occupy a symmetrical-position OBCA' with respect to its 1st-position at OBCA, and OBCA' and OBCA will be symmetrically-situated with respect to the plane of symmetry β.

If we considered the tetrahedron OBCA as a pyramid OBC-A, then a 90° rotation of the vertex-point A to its position at A" will take the pyramid OBC-A to its 2nd-position at OBC-A", and a 2nd 90° rotation of A at A" to its final-position at A' will take the pyramid OBC-A at OBC-A" to its final-position at OBC-A'.

49. THEOREMS ON SYMMETRY WITH RESPECT TO A POINT, LINE, OR HYPERPLANE.

Theorem 1. Any 2 figures symmetrically-situated with respect to a plane can be made to coincide point for point by a rotation of 1 of them through 180° around this plane as axis-plane.

Theorem 2. 2 figures symmetrically-situated with respect to a point can be made to coincide by a rotation of 1 of them through 180° around each of 2 absolutely-perpendicular planes through the point.

Theorem 3. 2 figures symmetrically-situated with respect to a line c can be put into positions of symmetry with respect to any hyperplane containing c by a rotation of 1 of them through 180° around the plane perpendicular to this hyperplane along c.

Theorem 4. 2 figures symmetrically-situated with respect to a hyperplane will not lose this relation of symmetry, if they are rotated around any plane of the hyperplane through the same angle in opposite-directions.

Theorem 5. If 2 figures are symmetrically-situated with respect to a hyperplane we can bring any 4 non-coplanar points of one into coincidence with the corresponding points of the other, in the hyperplane of symmetry, without disturbing this relation of symmetry.

Theorem 6. If 2 figures are symmetrically-situated with respect to a point, line, plane, or hyperplane, any segment of one is of the same-length as the corresponding segment of the other, and any 2 corresponding angles, dihedral-angles, or hyperplane-angles are equal.

Reference for the proofs of these theorems can be found in Manning's Geometry of Four Dimensions, Arts. 83-87, pp. 146-153.

III. ISOCLINE-PLANES

50. RECTANGULAR-SYSTEMS USED IN STUDYING THE ANGLES OF 2 PLANES. In this section, as in the 3rd-section of Chap. IV (Art. 38), we shall use the language of Point-Geometry, all lines, planes, and hyperplanes being assumed to pass through a given point O.
This section will be a continuation of the study of the isocline-planes of the Point-Geometry.

When 2 planes α and β are isocline to one another, the angles ϕ and ϕ' which α and β cut-out on their common-perpendicular planes can be any angles whatever, positive or negative.
With a particular plane α (of order pq) each of the angles ϕ and ϕ' can be changed by any multiples of 2π, or both at the same-time by multiples of π.
When $\phi' = \phi$, α is isocline to the planes bc and ad; and by giving different values to ϕ we have an infinite-number of planes isocline to bc and ad, and to one another. These planes are all perpendicular to ba and cd, and constitute what is called a SERIES OF ISOCLINE-PLANES. (see Fig. 52, and logic-diag. 1)

51. THE COMMON-PERPENDICULAR PLANES OF AN ISOCLINE-SERIES. CONJUGATE-SERIES. When the plane α is isocline to bc, these 2 planes have an infinite-number of common-perpendicular planes on which they cut-out the same-angle ϕ, and any 2 of the common-perpendicular planes cut-out the same-angle on α as on bc (Art. 38).
As an example of this (refer to logic-diagram 2), suppose in the planes bc and α we lay-off an angle ϕ from b and p, the terminal $\frac{1}{2}$-lines r and u of these angles will themselves form an angle ϕ and will determine a plane γ perpendicular to the plane bc

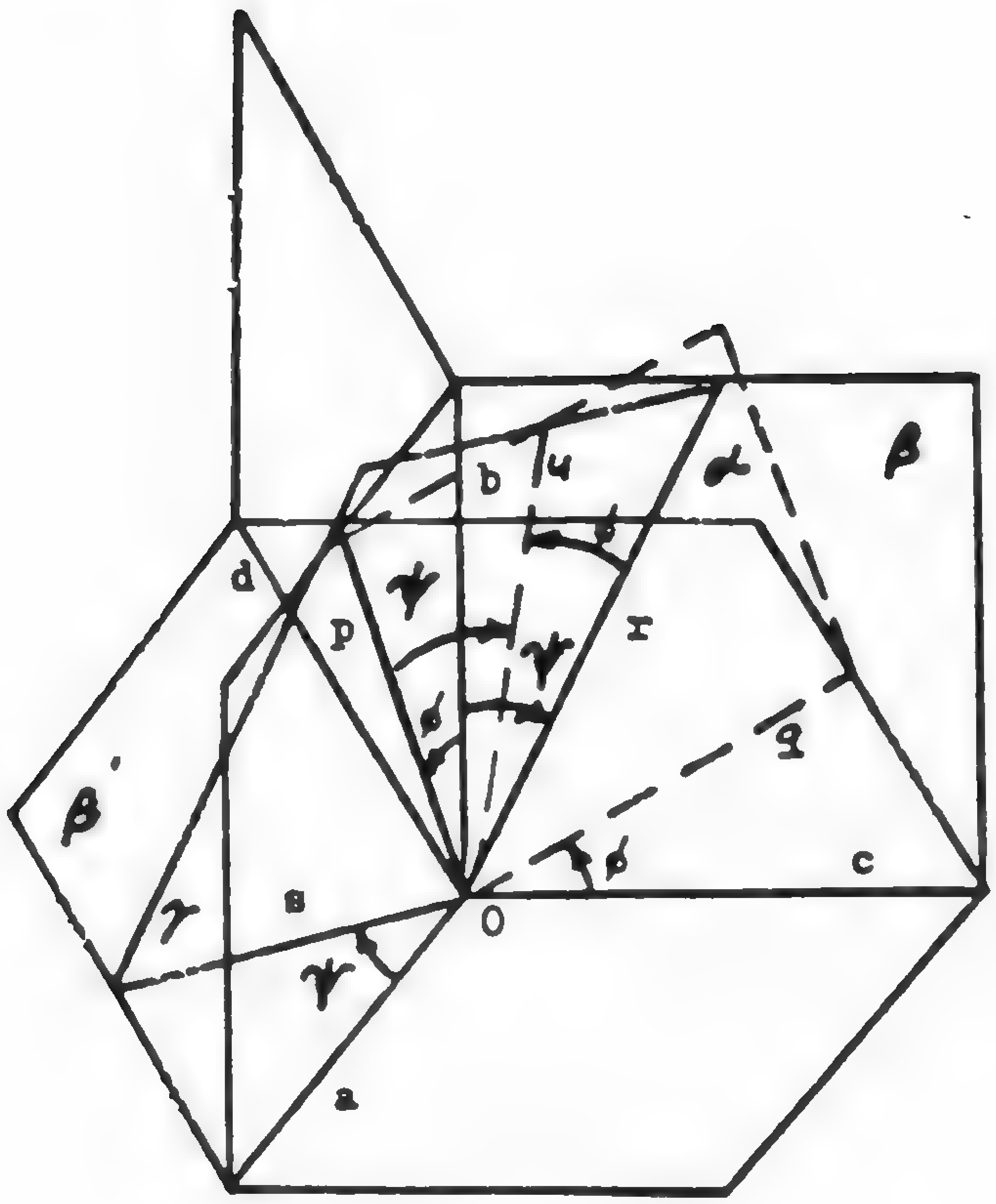
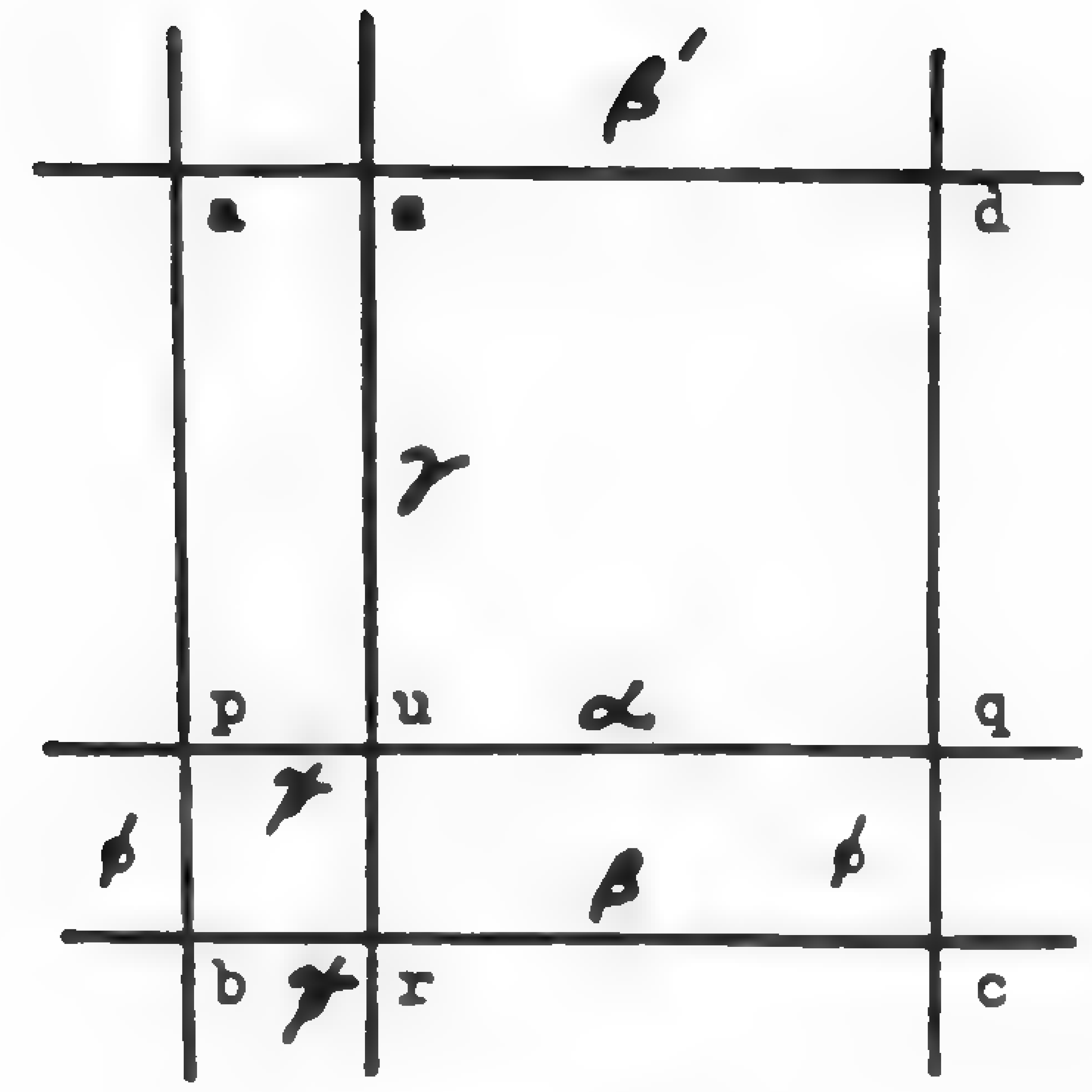


Fig. 59.



Logic-Diag. 2.

and α (~~we lay off an angle ϕ from b and p, the terminal $\frac{1}{2}$-lines r and u of these angles~~ (in ~~will themselves form an angle ϕ and will determine a plane γ perpendicular to the plane bc and to α~~)—it should be understood that the length of a line-segment in the logic-diagram represents an angle; for example, the line-segment br represents the angle ψ, and the $\frac{1}{2}$-line b is called the initial-side of ψ and the $\frac{1}{2}$-line r is called the terminal-side of the angle ψ.

Now γ, being perpendicular to bc, is perpendicular to its absolutely-perpendicular plane ad, and therefore is 1 of the common-perpendicular planes of α and ad. The terminal $\frac{1}{2}$-lines r and u of the angles ψ, forming in γ an angle ϕ, determine in γ a sense of rotation corresponding to bc; and the $\frac{1}{2}$-line s of γ, making an angle of $+90^{\circ}$ with the terminal $\frac{1}{2}$-line r which lies in bc, will be a terminal $\frac{1}{2}$-line s of ad, making with a the angle ψ in the same way that this angle was formed in the planes bc and α. The plane γ may, then, be regarded as determined by the angles equal to ψ laid-off on the planes bc and ad. But this construction is independent of the angle ϕ and the position of α. The plane γ is therefore perpendicular to all the planes of the isocline-series obtained by giving different values to ϕ and laying-off these angles in ba and cd from b and c.

By giving different values to ψ we have an infinite-number of planes γ perpendicular to all the planes α of the isocline-series. Starting with ba, we construct these planes in the same way that the α-series was constructed, and so they themselves form an isocline-series with the planes of the α-series for their common-perpendicular planes, each plane α perpendicular to all of them.

Thus we have associated with a rectangular-system, 2 series of isocline-planes, each plane of either series perpendicular to all the planes of the other series. We shall call them CONJUGATE-SERIES OF ISOCLINE-PLANES.

The planes α are not the only planes (through O) which are isocline to bc. We can rotate the rectangular-system around bc as an axis-plane, the $\frac{1}{2}$-lines a and d rotating in ad through any angle to new positions, and in the new rectangular-system we can construct a new-series of planes isocline to bc and ad, with a new-series of common-perpendicular planes, perpendicular to all of them but not perpendicular to any plane of the 1st-series except to bc and to ad themselves.

Fig. 59 is the graphic-form associated with logic-diag. 2.

In the planes bc and ad lay-off an angle ψ, the terminal $\frac{1}{2}$-lines r and s of these angles will determine a plane γ perpendicular to bc and ad and to all the planes of the α-series..In the plane pq lay-off an angle ψ, and let the terminal $\frac{1}{2}$-line of this angle be u, then the angle determined by the terminal $\frac{1}{2}$-lines r and u will be an angle ϕ, with u being the terminal $\frac{1}{2}$-line of this angle. The terminal $\frac{1}{2}$-line u lies in the intersection of the planes pq and rs.~~(The common-perpendicular plane γ will cut-out the same angle ϕ on all the planes of the α-series of isocline-planes.)~~

It should be noted that the angles ψ in the graphic are generated ~~in~~ a clockwise-direction around the point O in the planes bc and ad, and in the plane α from the $\frac{1}{2}$-line p in the direction towards the $\frac{1}{2}$-line q. The angles ϕ are generated in a counterclockwise -direction around O in the planes ba and cd, and in the plane γ from the $\frac{1}{2}$-line r in the direction towards the $\frac{1}{2}$-line s.

In the graphic-form of the isocline-planes of the Point-Geometry, we could consider that portion of the planes formed by any 2 of the 4 mutually-perpendicular $\frac{1}{2}$-lines at a point O of a rectangular-system as QUADRANTAL-PLANES; that is, for the positive-portion of a rectangular-system O-abcd.

When we make a study of the hypersphere, we shall see that the logic-diagram associated with it represents 1/16 of a hypersphere; that is, the positive-portion of the hypersphere for a given rectangular-system at a point O of the Point-Geometry associated with the hypersphere. The positive-portion of the rectangular-system is called the principle HEXADEKANT, thus, we would have an exact 1-to-1 correspondence between the graphic-form of the Point-Geometry and the logic-diagram. We could then work directly with the graphic-form for other portions of the quadrantal-planes about the point O. In fact, since the logic-diagram represents that portion of the hypersphere lying in 1 hexadekant, we would need 16 logic-diagrams to completely represent the entire hypershere; that is, 16 hexadekants, and in each hexadekant a portion of the hypersphere, or 1/16-hypersphere.

52. THE 2 SENSES IN WHICH PLANES CAN BE ISOCLINE. CONJUGATE-SERIES ISOCLINE IN THE OPPOSITE-SENSE. There are 2 senses in which a plane can be isocline to a given plane corresponding to the 2 possible arrangements of a rectangular-system. With a given rectangular-system, using the construction of Art. 38, we can say that the plane α is isocline to bc in one-sense when we make $\phi' = \phi$, and in the opposite-sense when we make $\phi' = -\phi$.

Starting with a plane β and a pair of absolutely-perpendicular planes γ and γ' perpendicular to β , let b and c be $\frac{1}{2}$-lines common to β and to γ and γ' respectively. If we lay-off 2 angles in the same-direction from b in the plane γ and in the same-direction from c in γ', or if we lay-off 2 angles in opposite-directions from b in γ and in the opposite-direction from c in γ', we shall have 2 planes isocline to β in the same-sense. But if we take the same-direction in 1 of the 2 perpendicular planes and opposite-directions in the other, we shall get 2 planes isocline to β in the opposite-senses.

When 2 planes are isocline to a given plane in opposite-senses we can speak of one as POSITIVELY-ISOCLINE and the other as NEGATIVELY-ISOCLINE.

If α is the plane pq of Art. 38, and is isocline to bc, we can determine the sense in which it is isocline by considering the order of 4 $\frac{1}{2}$-lines b, c, p, and q. Now in this determination we can take in each plane, instead of the 2 given $\frac{1}{2}$-lines, any 2 non-opposite $\frac{1}{2}$-lines, determinining their order by a positive-rotation of less than 180°. That is, if p' and q' are 2 $\frac{1}{2}$-lines in the plane pq such that a positive-rotation of less than 180° turns p' to the position of q', we shall have order bcp'q' = order bcpq; for p and q can be turned to the positions of p' and q' without becoming opposite, and so without changing this order (By a theorem of the 3-dimensional geometry, which states that order in a hyperplane is unchanged by any motion in the hyperplane.). In the same-way we can take for b and c any 2 non-opposite $\frac{1}{2}$-lines in the plane bc such that a positive-rotation of less than 180° will turn the 1st to the position of the 2nd. Conversely, we can determine the order of 4 non-coplanar $\frac{1}{2}$-lines drawn from O with reference to the order of 2 isocline-planes or any 2 planes which have only the point O in common.

Theorem 1. If α is isocline to β , β will be isocline to α in the same-sense.

Theorem 2. 2 conjugate-series of isocline-planes are isocline in opposite-senses.

2 absolutely-perpendicular planes are isocline in both senses, but in only 1 sense when we distinguish in each a particular direction of rotation. Thus the rectangular-system ad and da are isocline to bc in opposite-senses.

53. PLANES THROUGH ANY LINE ISOCLINE TO A GIVEN PLANE. PLANES TO WHICH INTERSECTING PLANES ARE ISOCLINE.

Theorem 1. Through any $\frac{1}{2}$-line not in a given plane nor perpendicular to it 2 planes can be passed isocline, one positively and the other negatively, to the given plane. (Fig. 60, and Logic-Diag. 3)

Given: A $\frac{1}{2}$-line p not in a plane β nor $\perp$ to it.

To Prove: 2 planes can be passed isocline, one positively and the other negatively, to β.

Proof: If we pass a plane through p $\perp$ to β, we can determine a rectangular-system with 4 mutually-perpendicular $\frac{1}{2}$-lines, a, b, c, and d, and so taken that β is the plane bc and p a $\frac{1}{2}$-line in the interior of the angle ba. Then we can take ϕ equal to the angle bp in the plane ba, and lay-off ϕ and $-\phi$ from c in the plane cd. In the plane cd, let cq and cq' be the angles formed by the $\frac{1}{2}$-lines q and q' with c, such that the angle cq equals ϕ and the angle cq' equals $-\phi$. The terminal $\frac{1}{2}$-lines q and q' of these angles determine with p 2 planes pq and pq', isocline to β in the 2 senses; that is, the plane pq is positively-isocline and the plane pq' is negatively-isocline. Therefore 2 planes can be passed isocline, one positively and the other negatively, to β. (Q.E.D)

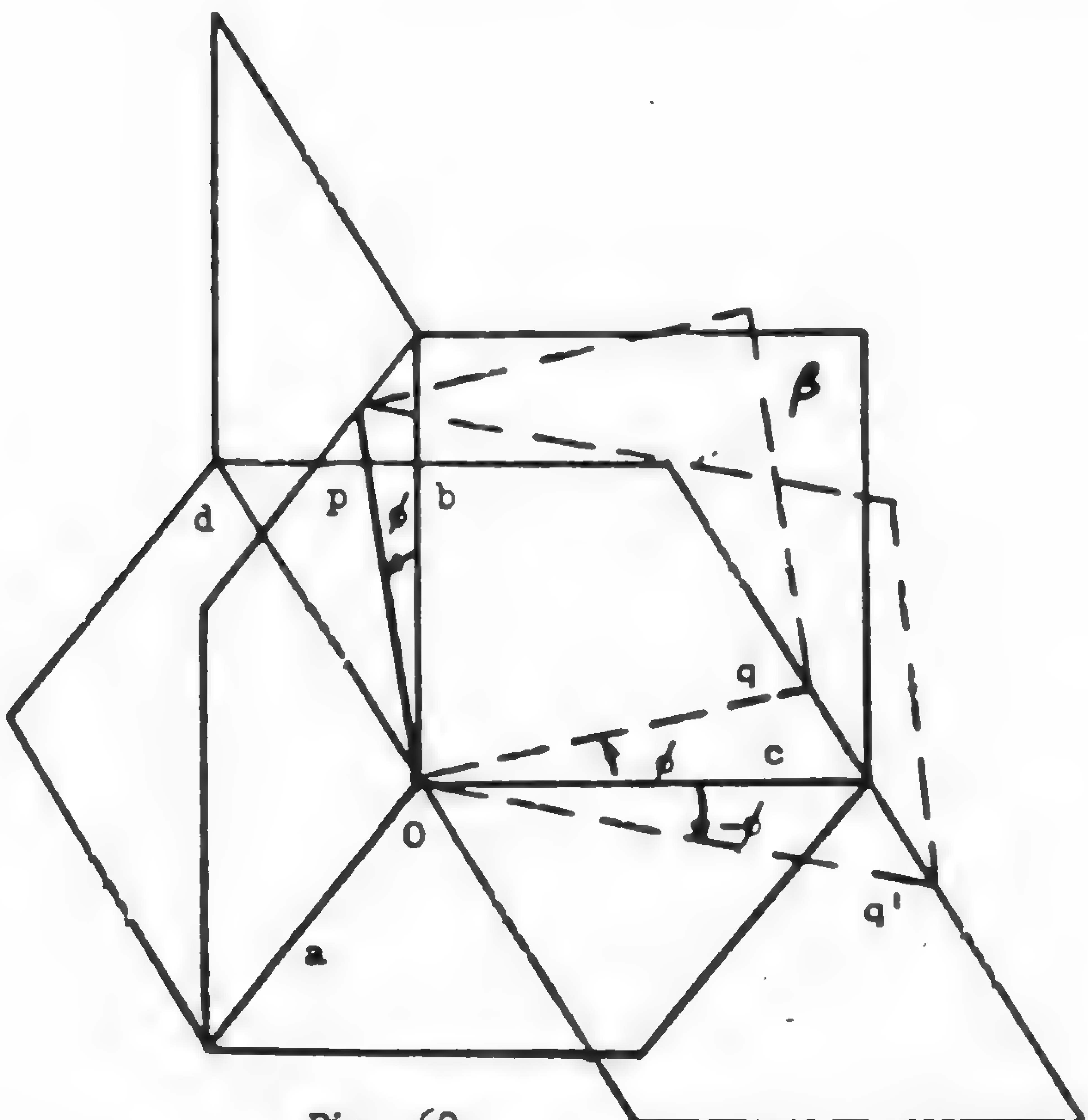
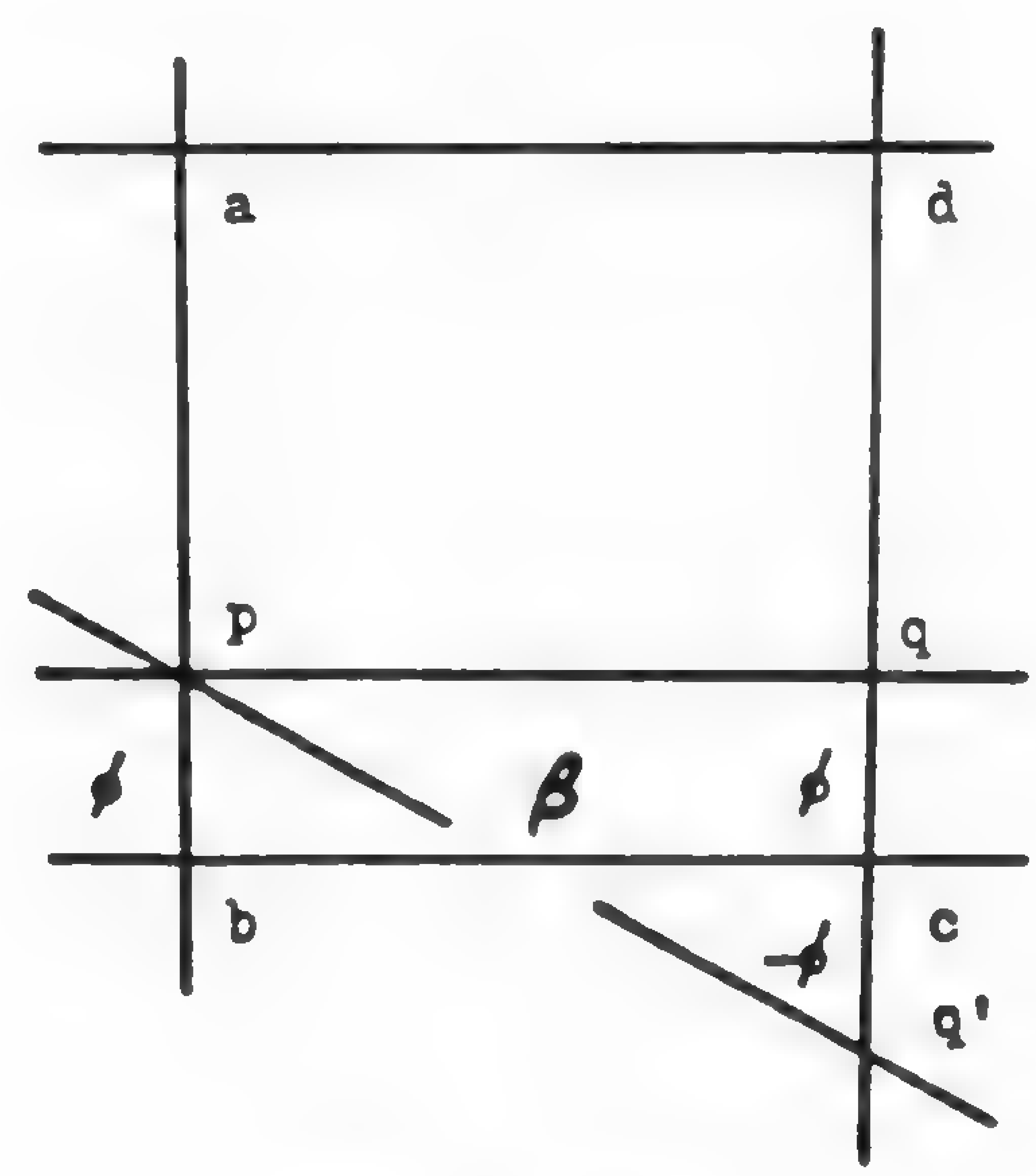


Fig. 60.



Logic-Diag. 3.

Theorem 2. 2 intersecting planes determine a pair of planes (absolutely-perpendicular to each other) to which they are isocline in one-way in the 2 senses respectively, and another pair to which they are isocline in the other-way in the 2 senses respectively. (Fig. 61, and Logic-Diag. 4)

Given: 2 planes α and β which intersect in a line PP'.

To Prove: α and β determine a pair of planes (Y to each other) to which they are isocline in one-way in the 2 senses respectively, and another pair to which they are isocline in the other-way in the 2 senses respectively.

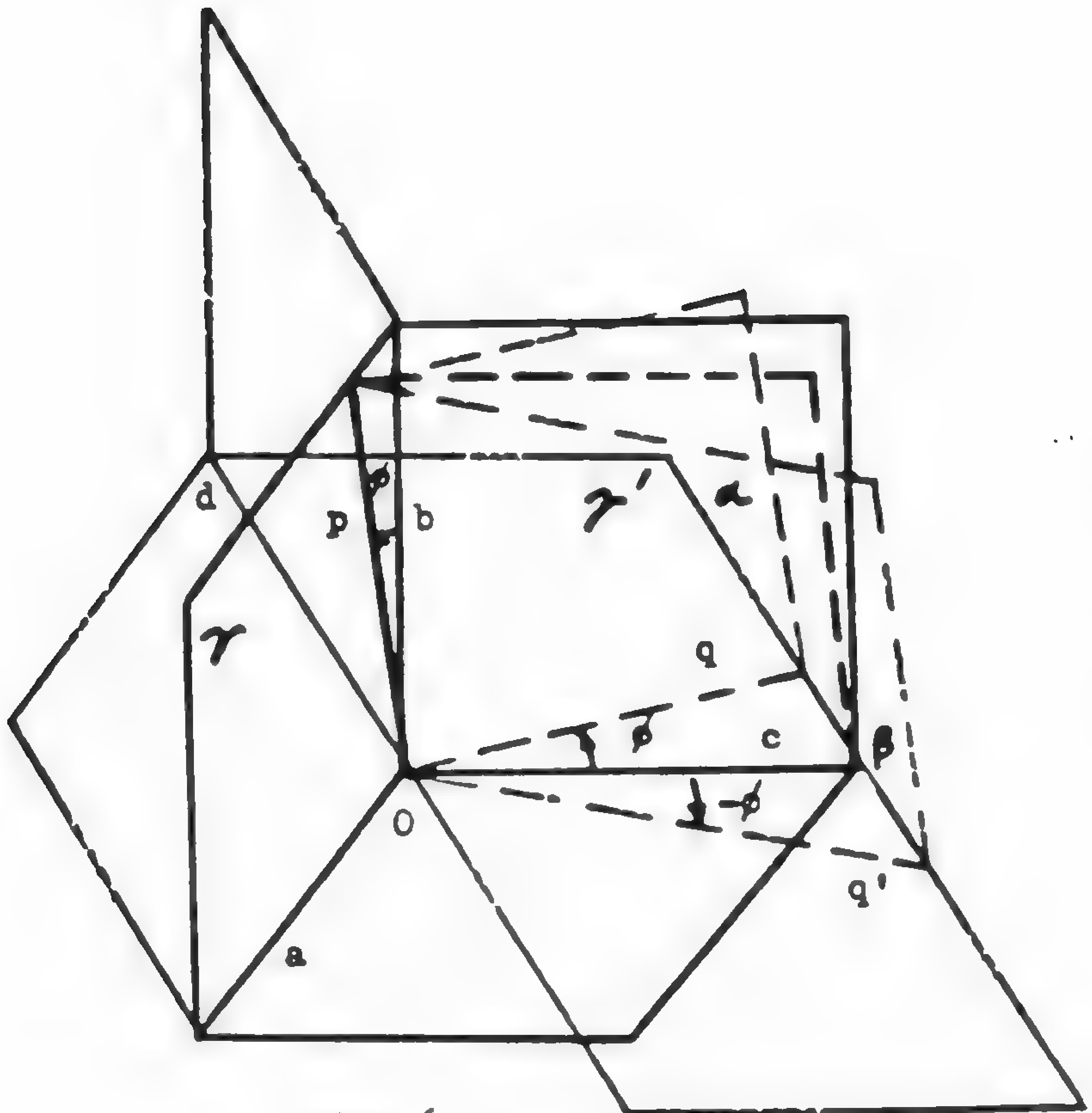
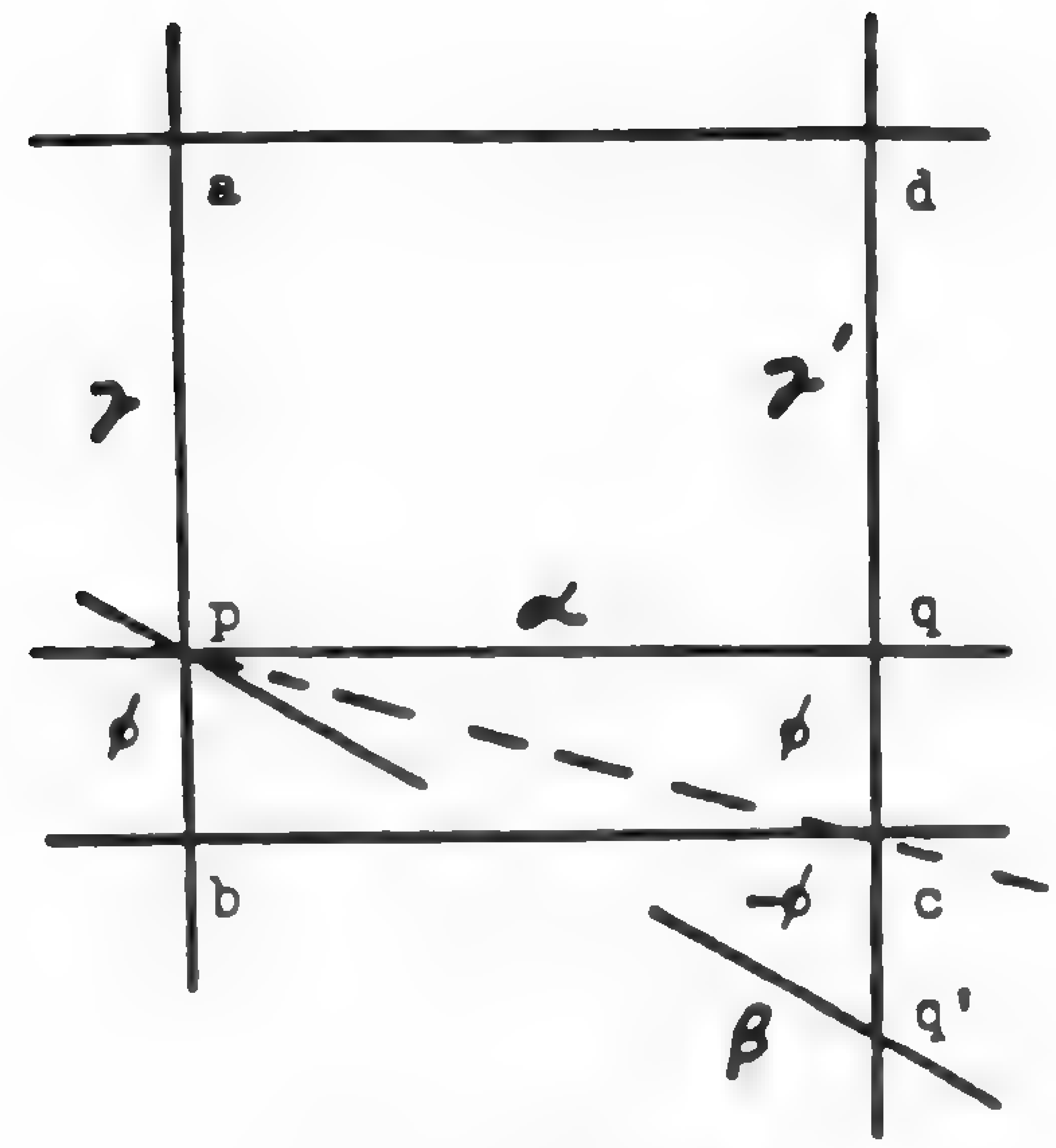


Fig. 61.



Logic-Diag. 4.

Proof: Let p be 1 of the opposite ½-lines in which the given planes α and β intersect (1 of the opposite ½-lines of PP' at O will be OP, which we shall call p.), and let γ and γ' be their common $\perp$ planes, γ passing through the ½-line p, and γ' the plane of the plane-angles of the dihedral-angles which they form. Let q and q' be the ½-lines which form 1 of these plane-angles, and let c be the ½-line bisecting the angle qq'; then the plane pq determined by the ½-lines p and q will be the plane α , the plane pq' determined by the ½-lines p and q' will be the plane β, and the plane pc determined by the ½-lines p and c will bisect the dihedral-angle α-p-β formed of the planes α and β.

In γ and γ' we establish directions of positive-rotation. Then in γ' the ½-lines q and q' form with c angles which may be called ϕ and $-\phi$. If now in γ we take a ½-line b so that the angle bp shall be equal to ϕ, we shall have the plane bc to which α and β are isocline in the 2 senses, as also to its Y plane ad.

If, on the other hand, we take b so that the angle bp shall be equal to $-\phi$, we shall have another plane bc to which the given planes α and β are isocline in the 2 senses, as also to its Y plane ad. Therefore the theorem is proved. (Q.E.D)

Further developments of the theory of the isocline-planes of the Point-Geometry can be found in Manning's Geometry of Four Dimensions—Section VI, Arts. 108-112, pp. 188-198.

The graphic-forms for some of the more 'complex-theorems' of the isocline-planes is quite-involved and will not be given in this small-treatise. The most interesting theorems are those on 'Poles and Polar-Series, and Isocline-Rotation in hyperspace'.

We shall now give the graphic-forms and proofs of the theorem of Art. 37 and theorem 1 of Art. 38.

Theorem (of Art. 37). When 2 planes α and β do not intersect, the plane of the minimum-angle which a ½-line of α makes with β is perpendicular to α and β. (Fig. 62, and Logic-Diag. 5.)

Given: 2 planes α and β that do not intersect.

To Prove: The plane of the minimum-angle which a ½-line of α makes with β is $\perp$ to α and β.

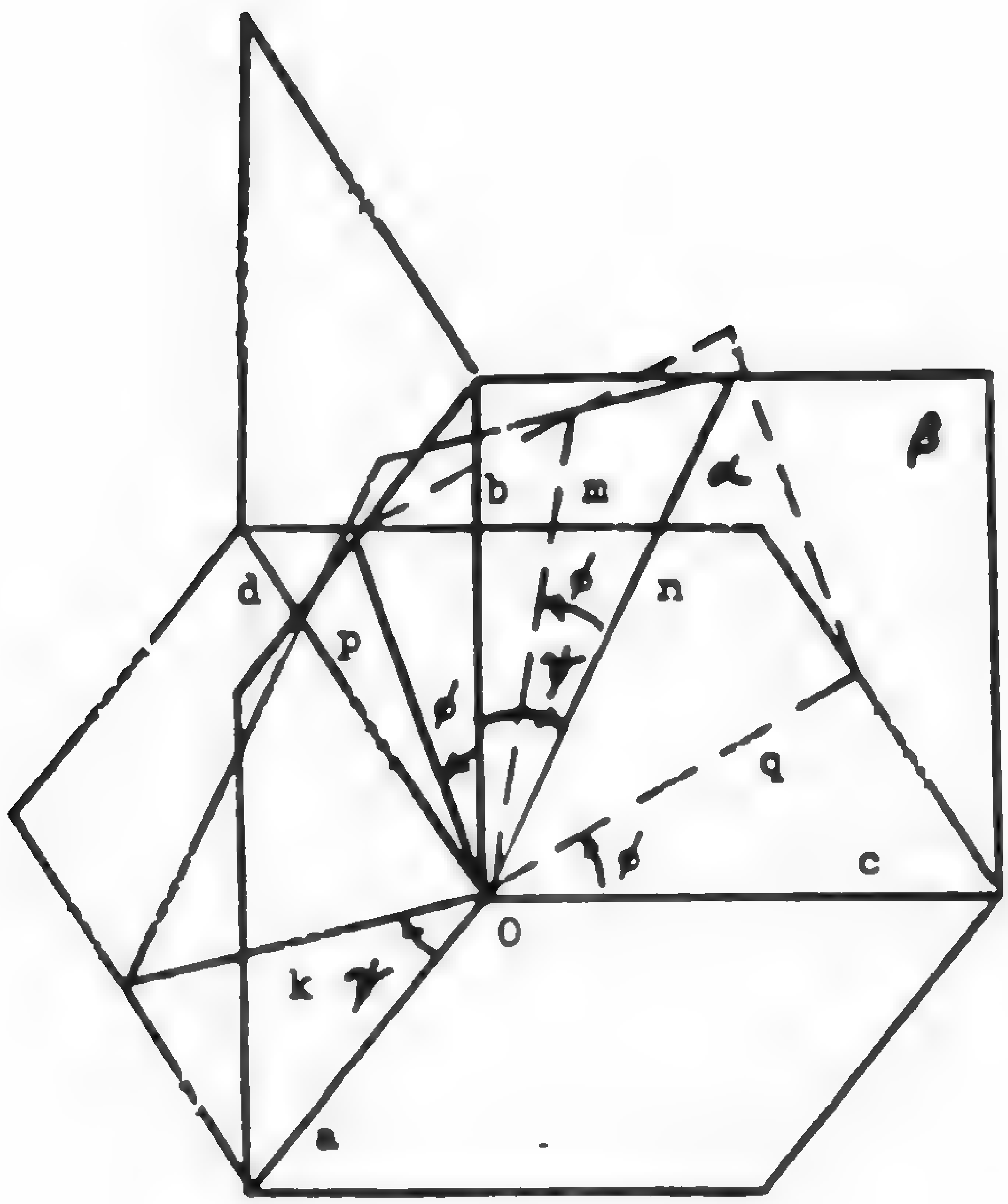
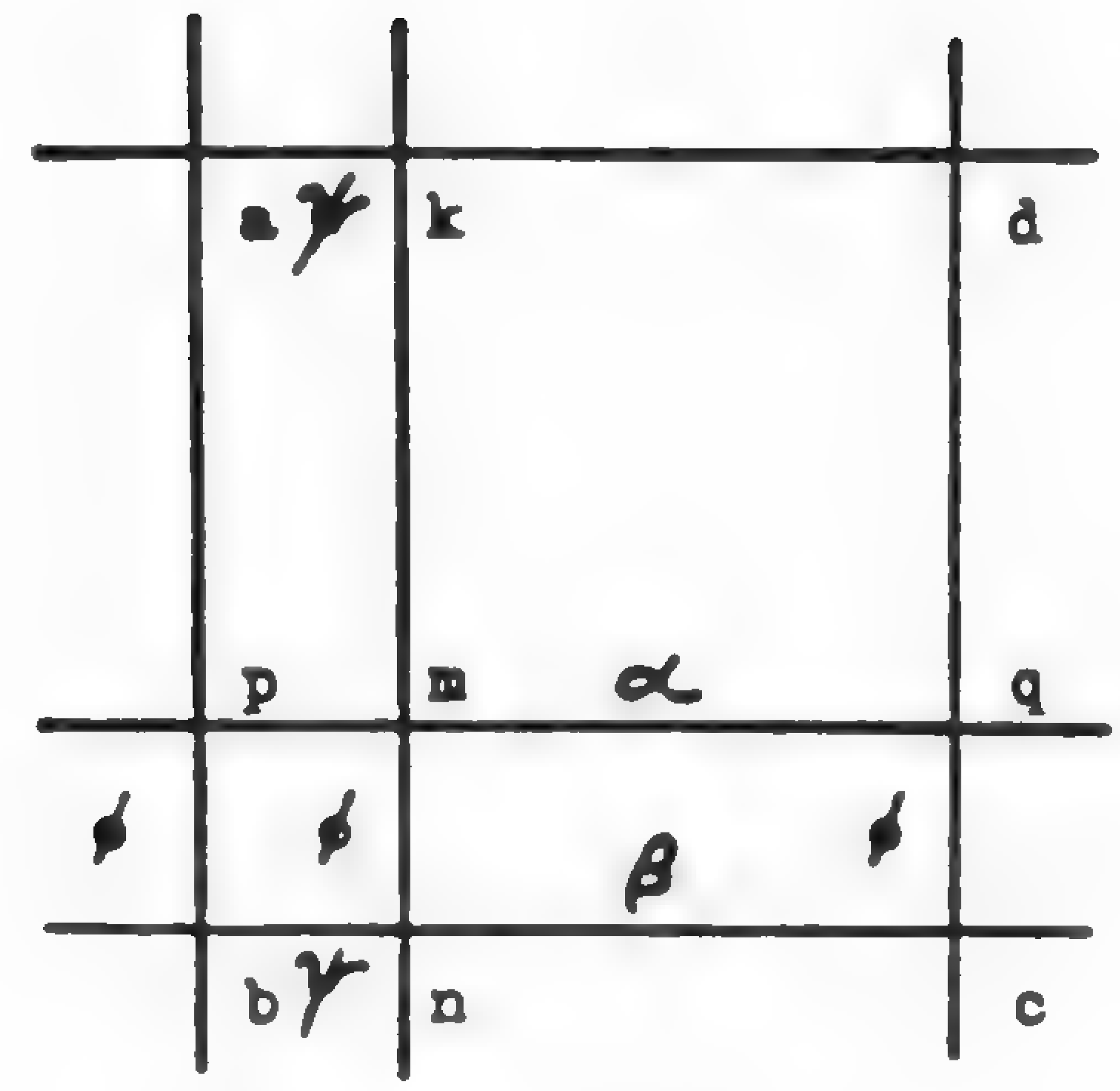


Fig. 62.



Logic-Diag. 5.

To resolve-out this theorem with a graphic-solution we shall make use of a portion of Logic-Diag. 5 and of Fig. 62. We construct a plane kn perpendicular to α and β which is also perpendicular to α at m; that is, the plane kn is perpendicular to α along a $\frac{1}{2}$-line m.

Proof: (Reductio ad absurdum.) Let m be a $\frac{1}{2}$-line of α which makes with β an angle ϕ, less than or equal to the angle made with β by any other $\frac{1}{2}$-line of α, and let n be the projection of m upon β, so that the plane mn is $\perp$ to β along n. If the plane mn is not $\perp$ to α, the projection of n upon α will be a $\frac{1}{2}$-line of α which makes with 1 $\frac{1}{2}$-line of β, and therefore contains with β, an angle less than ϕ. But this is contrary to hypothesis; and the plane mn must therefore be a common $\perp$ plane to α and β. Therefore the plane of the minimum-angle which a $\frac{1}{2}$-line of α makes with β is $\perp$ to α and β. (Q.E.D)

The plane mn (the common $\perp$ plane of the above theorem) intersects α and β in 2 pairs of opposite $\frac{1}{2}$-lines, and in the plane mn we have 2 pairs of vertical-angles, 1 pair of acute-angles equal to ϕ, and 1 pair of obtuse-angles (unless α and β are Y).

The plane Y to the plane mn is also $\perp$ to α and β (Art. 36, Th. 5). Let ϕ' be 1 of the acute-angles (or right-angles) lying in the intersection of this plane with α and β. We may let m' and n' be the $\frac{1}{2}$-lines forming the angle ϕ', m' in α and n' in β. n and n' (when ϕ' is not a right-angle) are the projections of m and m' upon β, and that portion of α which lies within a right-angle mm' will be projected upon that portion of β which lies within the right-angle nn'.

Theorem 1 (Art. 38). The angles which a plane makes with 1 of 2 absolutely-perpendicular planes are the complements of the angles which it makes with the other; and any 2 planes make the same angles as their absolutely-perpendicular planes. (Fig. 63, and Logic-Diag. 6.)

Case 1. Given: 2 Y planes α and α', and the angles ϕ and ϕ' that a plane β makes with α.

To Prove: The angles ϕ and ϕ' which a plane β makes with α are the complements of the angles that it makes with α'.

Proof: In a rectangular-system O-abcd, we can take for α the plane bc, α' the plane ad,

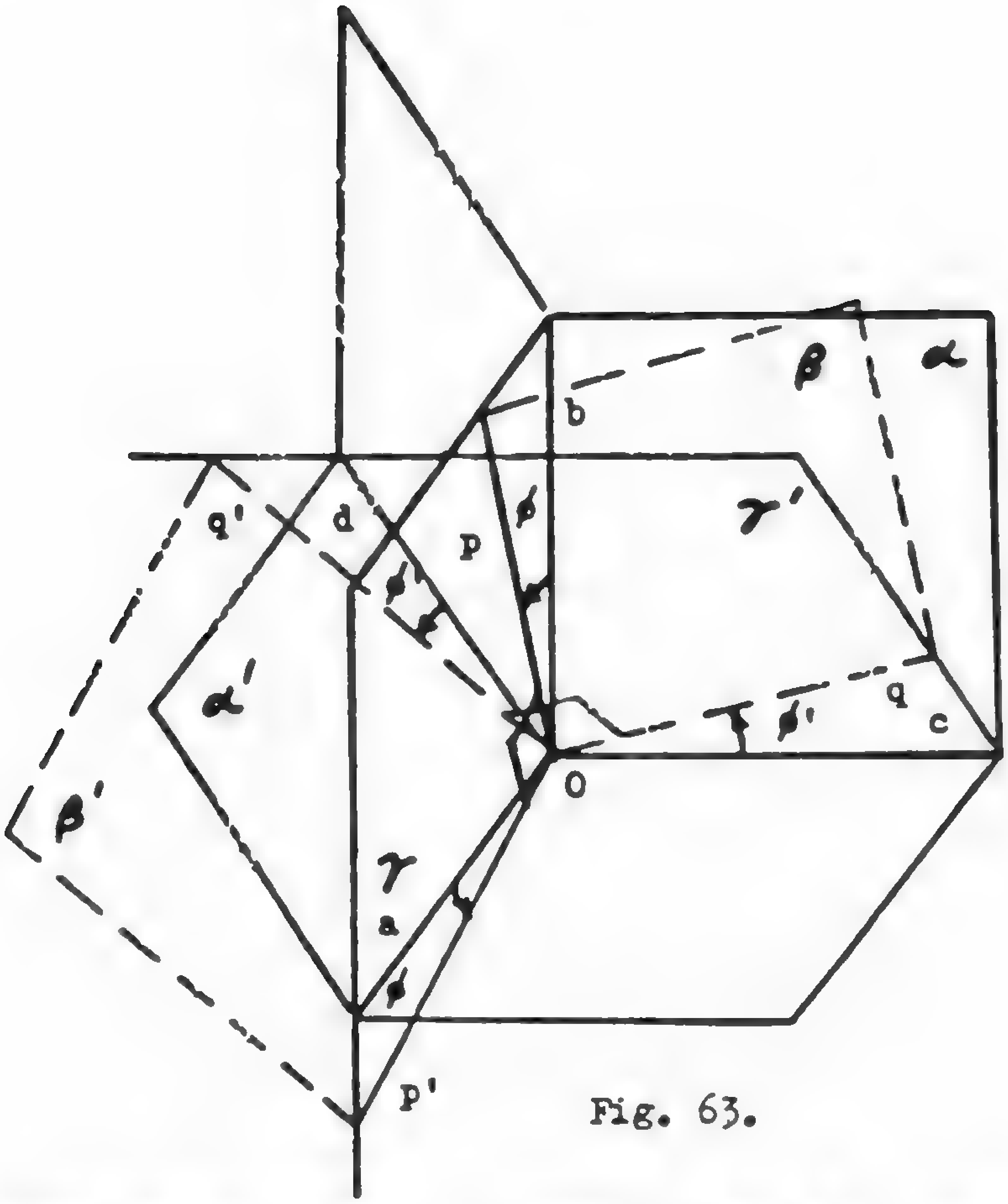
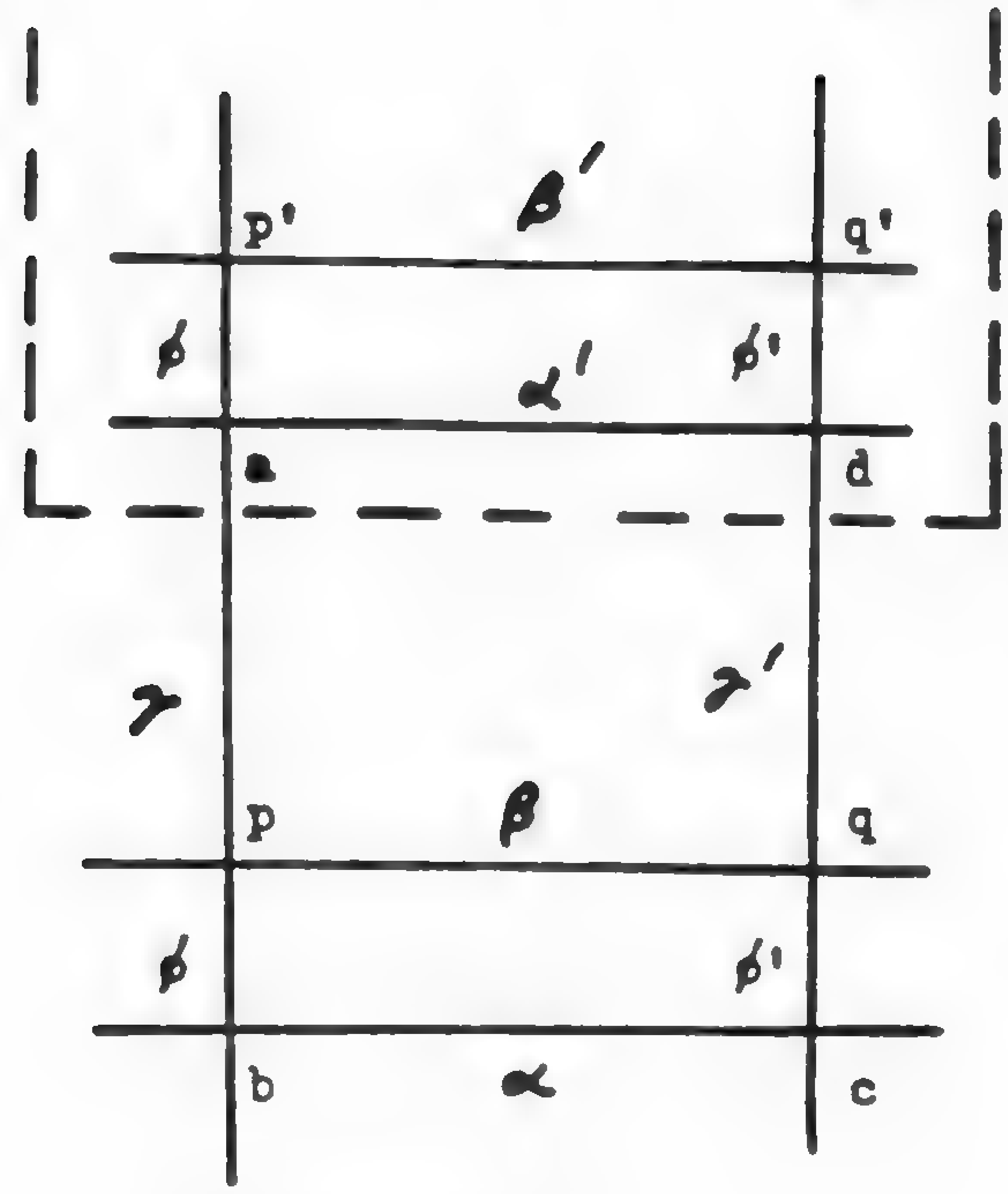


Fig. 63.



Logic-Diag. 6.

and for the common ⊥ planes of α and β , the planes ba and cd. The angles ϕ and ϕ' are laid-off in ba and cd respectively, and the terminal $\frac{1}{2}$-lines p and q of these angles determine the plane β . Now in the common ⊥ plane cd, the angle ϕ' is formed by the $\frac{1}{2}$-lines c and q, and the angle that β makes with α' is formed by the $\frac{1}{2}$-lines q and d. But cd is a right-angle, and therefore the sum of the angles cq and qd such that cq + qd = 90°, with qd = $90^\circ - \cancel{cq} = 90^\circ - \phi'$. Therefore the angle ϕ' is the complement of the angle qd. In the same way, then, the angles bp and pa formed in the other common ⊥ plane ba are the complements of each other; that is, ϕ is the complement of the angle pa that β makes with α'. Therefore the angles ϕ and ϕ' which a plane β makes with α are the complements of the angles that it makes with α'. (Q.E.D)

Case 2. Given: 2 planes α and β and their ⊥ planes α' and β', and the angles ϕ and ϕ' that β makes with α . (Use O-abcd as given in case 1.)

To Prove: The angles ϕ and ϕ' that β makes with α are the same as the angles that β' makes with α'.

Proof: In the common ⊥ plane cd, rotate the $\frac{1}{2}$-line q of 90° (counterclockwise) around the point O to the position of q', and in the other common ⊥ plane ba, rotate the $\frac{1}{2}$-line p of β 90° to the position p', then the $\frac{1}{2}$-lines p' and q' will determine the plane β' ⊥ to β . Now ba and cd are also the common ⊥ planes to α' and β', and the angles between α' and β' are formed in the same way that the angles ϕ and ϕ' were formed between α and β ; that is, in ba, the angle ap' will be 1 of the angles between α' and β', and this angle must be the same as ϕ, for the angle pp' being a right-angle, we can form the sum of the angles pa and ap', with pa + ap' = 90°, and the right-angle ba becomes the sum of the angles bp and pa, with bp + pa = 90°; and therefore pa + ap' = bp + pa, and cancelling pa on both sides of this equation leads to bp = ap', or ϕ = ap'. In the same way, then, we form the angle ϕ' between α' and β' in the other common ⊥ plane cd, and therefore, angle cq = dq', or ϕ' = dq'. Therefore the angles ϕ and ϕ' that β makes with α are the same as the angles that β' makes with α'. (Q.E.D)

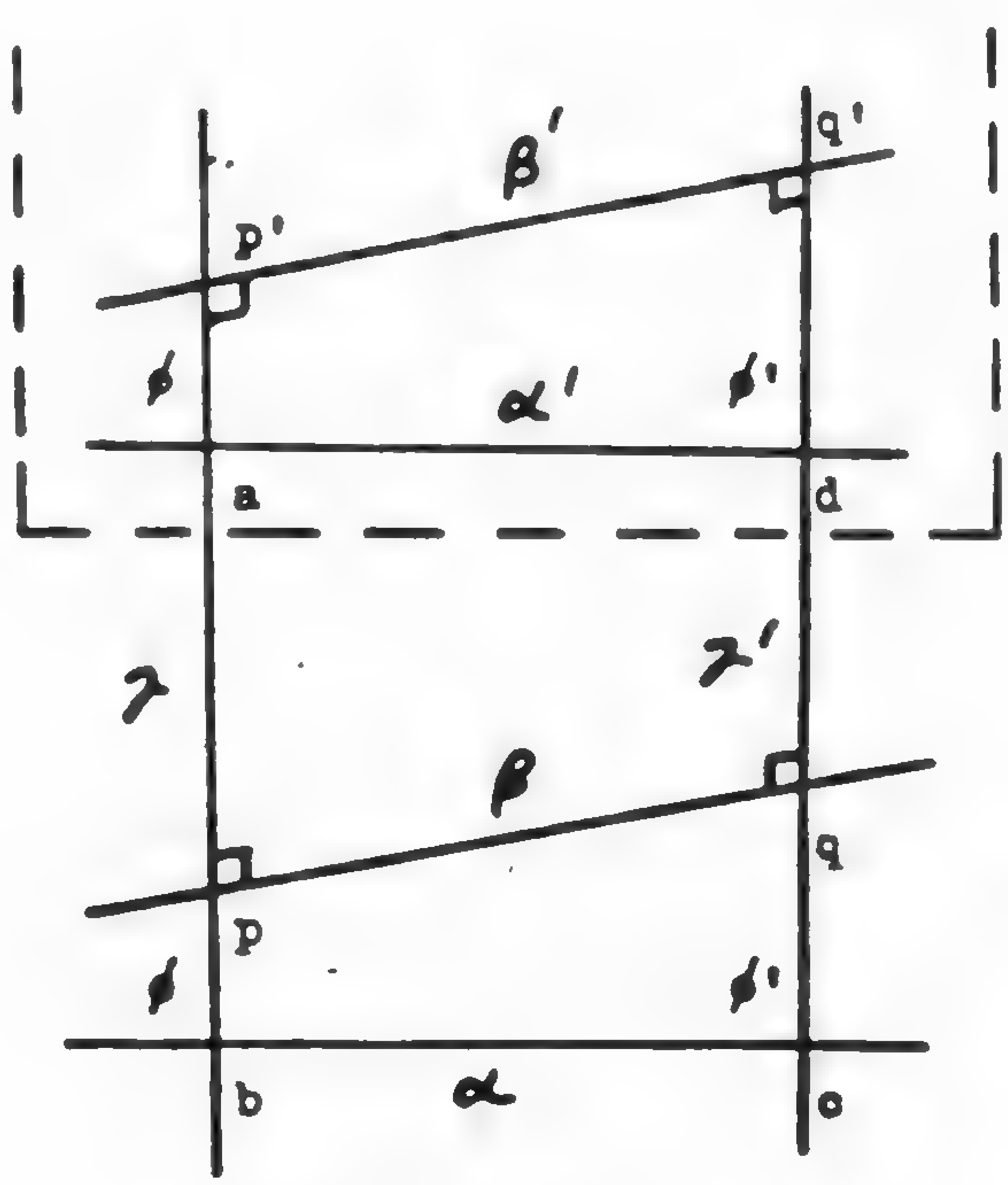
Referring to Logic-Diag. 6, the student should note that the absolutely-perpendicular planes β and β' also form a rectangular-system O-pqp'q'. The dashed-opened-partition forms a partial-block for another rectangular-system O-ad..; that is, the plane absolutely-perpendicular to α' will be α, but in fact, the logio-diagram represents quadrantal-planes, and so the quadrantal-plane ad if rotated 90° to the position -(bc), or b'c' (opposite $\frac{1}{2}$-lines of b and c) will ~~again~~ be absolutely-perpendicular to α', and therefore absolutely-perpendicular to its quadrantal-plane +(ad).

In logio-diag. 7, the planes pq and p'q' are isocline to each other, since the angles that they cut-out on their common-perpendicular planes ba and cd are equal, and in this case, are equal to 90°. The logio-diagrams can be used in numerous-ways. The student should become aware that the dihedral-angles at the corners of the 'parallelogram' pqp'q' are right-angles, since any plane intersecting 2 absolutely-perpendicular planes will be perpendicular to both. Care must be used in interpreting the logio-diagrams. Considerable knowledge of the Point-Geometry can be obtained by their use, in fact, without their use, the graphic-forms of the Point-Geometry would have come about slowly into existence, and the discovery of its geometric-properties, highly-improbable.

Logio-Diag. 6 is a combination of logic-diagrams: that is, 1 complete-diagram and a partial-diagram.

Since we have already covered the basic-development of isocline-planes we will repeat here some of its basic-properties.

When the angles of 2 planes are equal the planes have an infinite-number of common-perpendicular planes, but they cut-out the same-angles on them all. The 2 planes are then said to be ISOCLINE. Absolutely-perpendicular planes are always isocline, and a plane isocline to 1 of 2 absolutely-perpendicular planes is isocline to the other. Any 2 lines taken 1 in each of 2 absolutely-perpendicular planes determine a common-perpendicular plane (Art. 36, Th. 4), but in the case of 2 isocline-planes which are not absolutely-perpendicular only 1 plane of the common-perpendicular planes passes through any line of either. Any 2 of these common-perpendicular planes cut-out equal angles on the 2 given planes and are themselves isocline.



Logio-Diag. 7.

CHAPTER VI

HYPERPYRAMIDS, HYPERCONES, AND THE HYPERSPHERE

I. PENTAHEDROIDS AND HYPERPYRAMIDS

54. THEOREMS ON PENTAHEDROIDS. THE CENTER OF GRAVITY (CENTROID).

Theorem 1. In a pentahedroid, if 2 of the tetrahedrons can be inscribed in spheres, the lines drawn through the centers of these spheres perpendicular to their hyperplanes lie in a plane; when they meet in a point this point is equidistant from the 5 vertices of the pentahedroid, the 5 tetrahedrons can all be inscribed in spheres, and the 5 lines drawn through the centers of these spheres perpendicular to their hyperplanes all pass through the same point.

Theorem 2. The ½-hyperplanes bisecting the 10 hyperplane-angles of a pentahedroid all pass through a point within the pentahedroid, a point equidistant from the hyperplanes of its 5 cells.

Theorem 3. The ½-lines drawn from the vertices of a pentahedroid through the centers of gravity of the opposite-cells meet in a point.

Theorem 4. If each of the edges of a pentahedroid is equal to the corresponding edge of a 2nd pentahedroid, when the 5 vertices of one are made to correspond in some order to the 5 vertices of the other, the pentahedroid will be congruent to or symmetrical.

For proofs of these theorems see Manning's Geometry of Four Dimensions—Arts. 113-114, pp. 199-202.

If we take a regular-tetrahedron and draw a line through its center perpendicular to its hyperplane, every point of this line will be equidistant from the 4 vertices of the pentahedroid, and if we take a point at a distance from the 4 vertices equal to 1 of the edges of the tetrahedron, we shall have a pentahedroid in which the 10 edges are all equal. When a pentahedroid has all of its edges equal it is called a REGULAR-PENTAHEDROID.

55. THE TERMS RIGHT AND REGULAR USED OF HYPERPYRAMIDS AND DOUBLE-PYRAMIDS. When the base of a hyperpyramid is the interior of a regular-tetrahedron, the interior of the segment formed consisting of the vertex and the center of the base is called the AXIS OF THE HYPERPYRAMID; and when the line containing the axis is perpendicular to the hyperplane of the base the hyperpyramid is REGULAR.

Theorem 1. In a regular-hyperpyramid the lateral-pyramids are equal regular-pyramids. The axis of any 1 of these lateral-pyramids is the hypothenuse of a right-triangle whose legs are the axis of the hyperpyramid and a radius of the sphere inscribed in the base.

The SLANT-HEIGHT OF A REGULAR-HYPERPYRAMID is the altitude of any 1 of the lateral-pyramids.

When the base of a double-pyramid is the interior of a regular-polygon, the interior of the triangle determined by the vertex-edge and the center of the base is called the AXIS-ELEMENT OF THE DOUBLE-PYRAMID; and when the plane of this triangle is absolutely-perpendicular to the plane of the base we have a RIGHT-DOUBLE-PYRAMID. A right-double-pyramid is ISOSCELES when the extremities of the vertex-edge are at the same distance from the plane of the base. Such a double-pyramid is also called REGULAR.

Theorem 2. In a right-double-pyramid (with a regular-base) the lateral-faces are congruent, and the 2 end-pyramids are regular. In a regular-double-pyramid the end-pyramids are congruent.

II. HYPERCONES AND DOUBLE-CONES

56. SPHERICAL-HYPERCONES AND RIGHT-HYPERCONES. A SPHERICAL-HYPERCONE is one whose base is the interior of a sphere. The AXIS OF A SPHERICAL-HYPERCONE is the interior of a segment consisting of the vertex and the center of the base. A RIGHT-SPHERICAL-HYPERCONE, or simply a RIGHT-HYPERCONE, is one whose axis lies in a line perpendicular to the hyperplane of the base.

A section of a spherical-hypercone by a hyperplane containing the vertex and any point

of the base is a circular-cone.

The spherical-hypercone in Fig. 64 may be denoted by V-S. The base of the spherical-hypercone V-S is the interior of the sphere S lying in the hyperplane of the black-sphere , and the point V is the vertex. The center of the sphere S is the point O. The axis of the spherical-hypercone is the interior of the segment consisting of the vertex V and the center of the sphere, and is the interior of the segment VO.

We shall consider only right-spherical-hypercones in this text. The spherical-hypercone V-S has its axis VO lying in a line perpendicular to the hyperplane of the base.

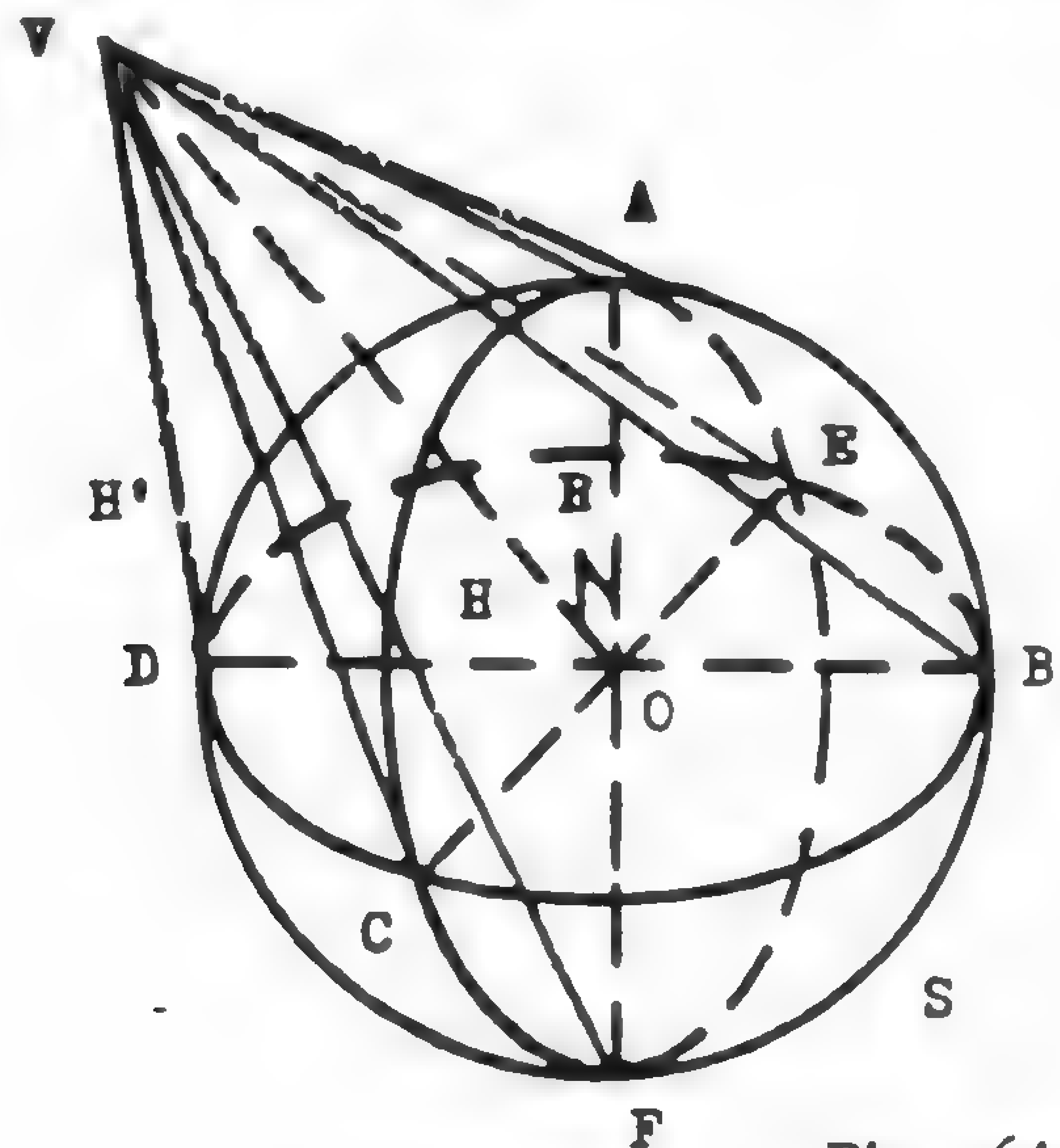


Fig. 64.

The section of the spherical-hypercone V-S by a hyperplane containing the vertex V and the center of the base will be a circular-cone. For example: V-(ACF), V-(BCD), V-(ABF), and so forth.

Theorem 1. When a right-triangle takes all possible positions with 1 leg fixed, the vertices and the points of the other 2 sides of the triangle make-up a right-spherical-hypercone. The fixed-side is the axis, the hypothenuse is an element, and the other leg is a radius of the base. (see Fig. 64.)

When a right-triangle AOV takes all possible positions with the leg VO fixed, the vertices A, O, and V and the points of the sides AO and AV make-up a right-spherical-hypercone V-S. The fixed-side VO is the axis, the hypothenuse VA is an element, and the leg OA is the radius R of the base.

Theorem 2. If in the hyperplane of a cone of revolution we pass a plane through its axis and rotate around this plane that portion of the cone which lies on one-side of it, we shall have all of a right-spherical-hypercone except that portion which makes-up the section of the cone by the plane. (Fig. 65.)

Let the hyperplane VACF be the hyperplane of a cone of revolution, then if we take the cone V-(ACF) as the cone of revolution in this hyperplane, and take the plane of the triangle VAF passing through its axis VO and rotate around the plane of this triangle as axis-plane the $\frac{1}{2}$-cone V-($\overline{ACF}$) which lies on one-side of it, we shall have all of a right-spherical-hypercone V-S except that portion which makes-up the section of the cone V-(ACF) by the plane of the triangle VAF.

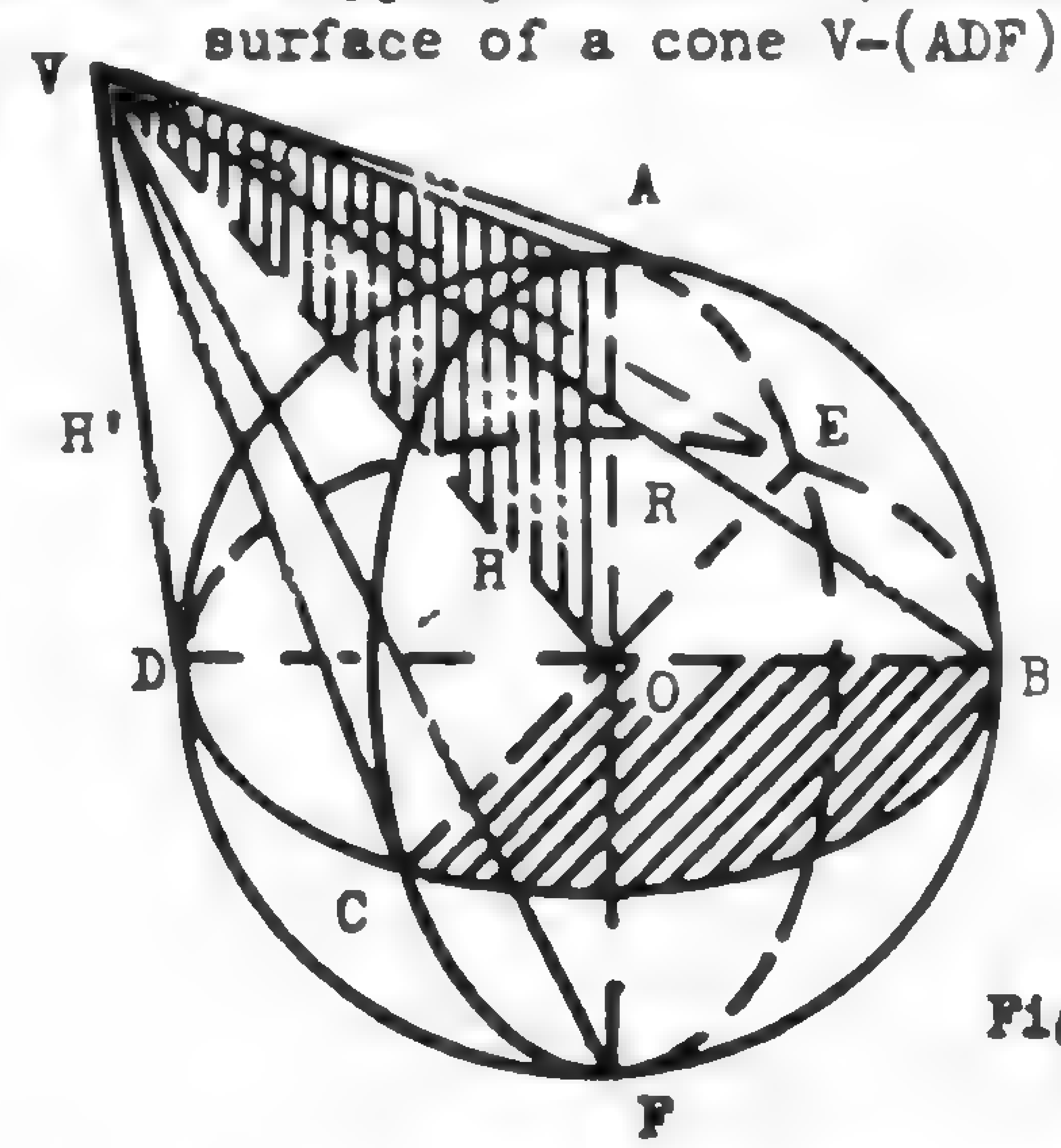
In other words, the plane $\perp$ to the plane of the triangle VAF at the point O will be the plane of the great-circle (BCD), and in the hyperplane of the sphere S, the point C of the $\frac{1}{2}$-cone V-($\overline{ACF}$) will rotate around the point O of AF in the plane of the great-circle (BCD), and therefore, the $\frac{1}{2}$-cone V-(ACF) will rotate around the plane of the triangle VAF as axis-plane (see Art. 47).

The hidden-views of the spherical-hypercone are easy to construct in the graphic-figure. But in order to make a study of the hidden-views of a spherical-hypercone, we need to make use of the $\frac{1}{2}$-spheres in the base of the spherical-hypercone. We shall make use of a special-notation for the $\frac{1}{2}$-spheres of a sphere as follows:

The symbol-combination V-S_A denotes a $\frac{1}{2}$-spherical-hypercone having the point V for its vertex, and whose $\frac{1}{2}$-base is a $\frac{1}{2}$-sphere S_A determined, in such-a-way, that the capital-letter A shall denote that portion of the $\frac{1}{2}$-sphere of S having A for 1 of its poles and with the spherical-edge (rim) of this $\frac{1}{2}$-sphere lying in the plane of the great-circle (BCD) perpendicular to the axis AF of S at O. The $\frac{1}{2}$-spherical-hypercone V-S_F will have the point F for pole of the $\frac{1}{2}$-sphere S_F and with the spherical-rim (BCD) of this $\frac{1}{2}$-sphere lying in the plane of the great-circle (BCD) perpendicular to AF at O, with the point V being the vertex of V-S_F. We have, then, $S = S_A + S_F$. In Fig. 65, the sphere S is partitioned into 6 $\frac{1}{2}$-spheres, i.e., $S_X = S_A, \ldots, S_F$.

The $\frac{1}{2}$-spherical-hypercone V-S_C will be a visible-view in the graphic-figure except that portion made-up by the interior of the $\frac{1}{2}$-sphere S_C. To see this, observe that in the hyperplane of the sphere S, the $\frac{1}{2}$-arc $\overline{DCB}$ lies on one-side of the plane of the great-circle (ABF) and the $\frac{1}{2}$-arc $\overline{BED}$ lies on the other-side of the plane of this great-circle, and therefore, the $\frac{1}{2}$-arc $\overline{DCB}$ lying on one-side of the plane of the great-circle (ABF) will lie on one-side of the hyperplane of the cone V-(ABF) and the $\frac{1}{2}$-arc $\overline{BED}$ will lie on the other side of the hyperplane of this cone. The $\frac{1}{2}$-arcs $\overline{DCB}$ and $\overline{BED}$ are generated by the rotation of the point D around the plane of the triangle VAF as axis-plane of the spherical-hypercone V-S, or in the hyperplane of the sphere S, around the axis AF in the plane of the great-circle (BCD) and around the point O lying in the plane of this great-circle.

We can generate the lateral-hypersurface of a $\frac{1}{2}$-spherical-hypercone by using a method somewhat similar to that of a double-cone. The vertex-points V, A, and D together with the edges VA and VD as well as the $\frac{1}{4}$-arc $\overline{AD}$ may be called a CONICAL-SECTOR-BOUNDARY. We shall denote a conical-sector by V-$\overline{AD}$. The face of V-$\overline{AD}$ is an element of V-S_A. The conical-face of V-$\overline{AD}$ lies in the hyperplane of a $\frac{1}{4}$-cone V-$\overline{(ADF)}$, and is therefore a portion of the lateral-surface of a cone V-(ADF).



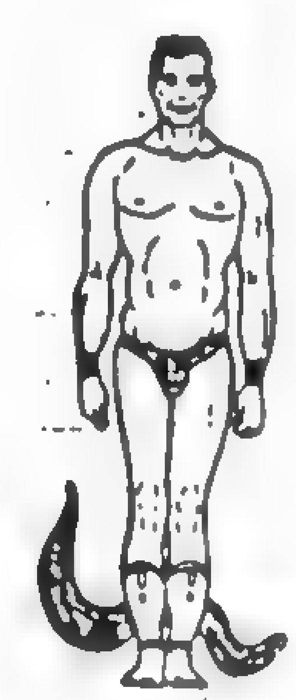


Fig. 65.

Considering the plane of the triangle VAF as axis-plane around which we rotate the conical-sector V-$\overline{AD}$, and taking the plane of the great-circle (BCD) $\perp$ to the given axis-plane at O, then we can let the edge VA of V-$\overline{AD}$ be the vertex-edge, and the great-circle (BCD) lying in the base-sphere S as a directing-circle, such that V-$\overline{AD}$ and the directing-circle (BCD) meet at the point D. Now if we rotate V-$\overline{AD}$ around the curved-path of the directing-circle (BCD) in either a clockwise or counterclockwise-direction from its position at D, with the vertex-edge VA fixed, we shall have all of the lateral-hypersurface of a $\frac{1}{2}$-spherical-hypercone V-S_A except that portion made-up of the vertex-edge VA and its 2 end-points V and A. We generate the lateral-hypersurface of the $\frac{1}{2}$-spherical-hypercone V-S_F using the same directing-circle (BCD), but with the vertex-edge VF and $\frac{1}{4}$-arc $\overline{DF}$ of the other conical-sector V-$\overline{DF}$.

In the rotation of V-$\overline{AD}$ around the plane of the triangle VAF, the edge VD will generate the lateral-surface of the cone V-(DCB), and the point D will generate the circle (DCB) enclosing the base of this cone. The $\frac{1}{4}$-arc $\overline{AD}$ will generate all of the surface of the $\frac{1}{2}$-sphere S_A except the pole A. We could also say that the point D lying on the $\frac{1}{4}$-arc $\overline{AD}$ will generate that portion of the $\frac{1}{2}$-sphere S_A's surface lying on the circle (DCB) and which we can call the EDGE of the $\frac{1}{2}$-sphere S_A.

The SLANT-HEIGHT OF A RIGHT-SPHERICAL-HYPERCONE is the distance from the vertex to any point of the sphere whose interior is the base; it is the length of the hypothemuse of the right-triangle of theorem 1. (see Fig. 64.)

H' is the slant-height of the spherical-hypercone V-S, and H is its altitude.

A study of the HYPERCONIC-SECTIONS of the hypersolid-geometry is a study in itself, and should prove quite illuminating to those interested in this part of the hypersolid-geometry, particularly in combination with the hypersolid-analytic-geometry.

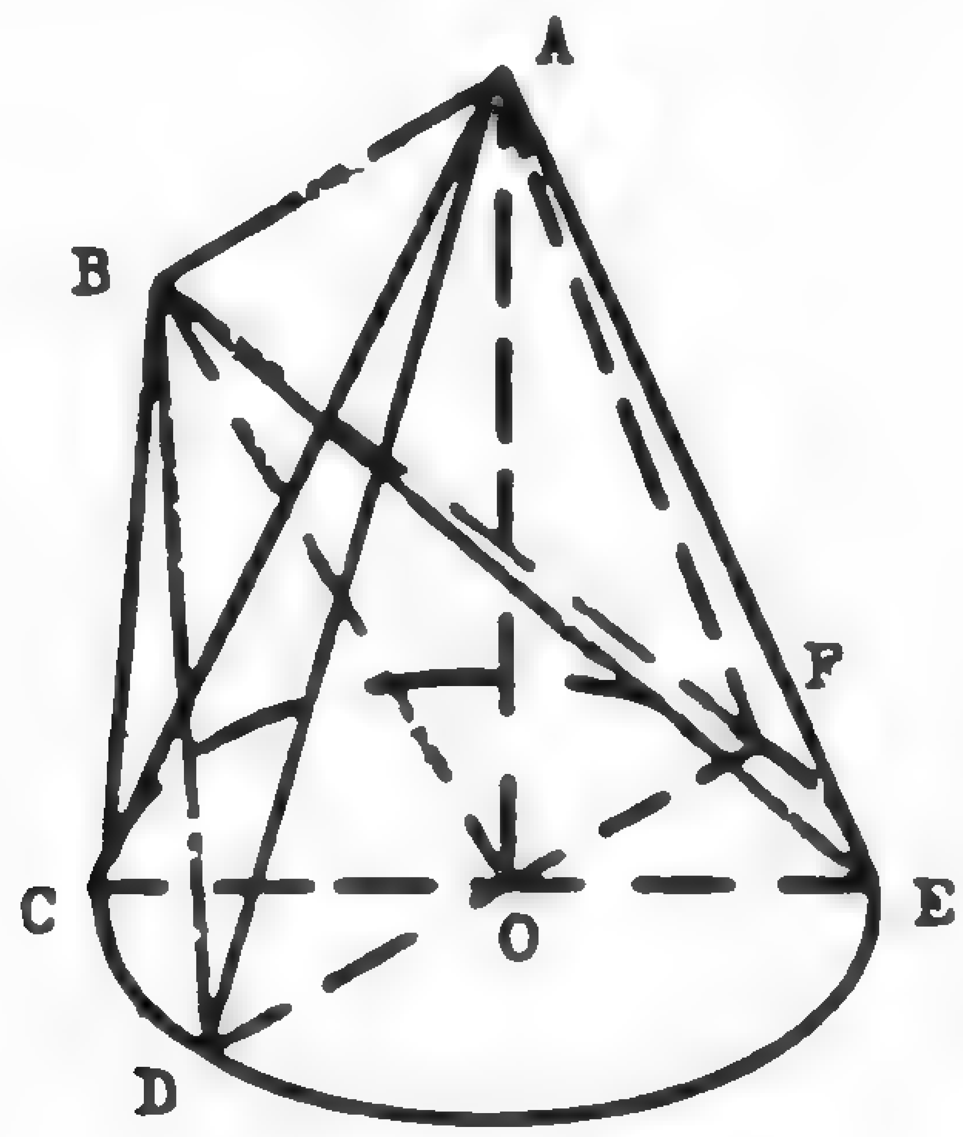


Fig. 66.

57. CIRCULAR-DOUBLE-CONES AND RIGHT-DOUBLE-CONES. A CIRCULAR-DOUBLE-CONE is one whose base is the interior of a circle. The AXIS-ELEMENT of a CIRCULAR-DOUBLE-CONE is the interior of the triangle determined by the vertex-edge and the center of the base. A RIGHT-CIRCULAR-DOUBLE-CONE, or simply a RIGHT-DOUBLE-CONE, is one whose axis-element lies in a plane absolutely-perpendicular to the plane of the base, and the double-cone is ISOSCELES when the extremities of the vertex-edge are at the same distance from the plane of the base.

We shall consider the double-cone AB-(CDE) in Fig. 66, an isosceles right-double-cone. Its base is the interior of the circle (CDE). The axis-element of AB-(CDE) is the interior of the triangle ABO determined by the vertex-edge AB and the point O at the center of the base; its axis-element lies in the plane of the triangle ABO $\underline{V}$ to the plane of the base (CDE), and the extremity-points A and B of the vertex-edge AB are at the same distance from the plane of the base (CDE), where AO = BO. The axis-element, or face of the triangle ABO, lies entirely in the interior of the double-cone AB-(CDE).

Theorem 1. In a right-double-cone the elements are congruent, and the 2 end-cones are cones of revolution. In an isosceles right-double-cone the end-cones are congruent. (Fig. 66.)

The 2 end-cones A-(CDE) and B-(CDE) are congruent and are cones of revolution.

Theorem 2. A right-double-cone may be generated by the rotation of a tetrahedron which has an edge and a face in a perpendicular line and plane, the rotation taking place around the latter. (see Fig. 66.)

If we take the tetrahedron ABCO with the line of the segment CO $\perp$ to the plane of the triangle ABO, then in a rotation around the plane of this triangle, the face ABC will generate the set of elements of the double-cone AB-(CDE), the face ABO will be the axis-element, the faces AOC and BOC will generate the interior of the 2 end-cones A-(CDE) and B-(CDE), and the edge CO will generate all of the base except the point O at its center.

If a tetrahedron has a rectangular-trihedral-angle at 1 of its vertices we can generate 3 different double-cones. The details of this are left for the student to work-out. For, consider a line perpendicular to the rectangular-trihedral-angle at its vertex, then we will have 3 pairs of absolutely-perpendicular planes, and with each pair of these

absolutely-perpendicular planes, we can determine a simple-rotation of the tetrahedron around 1 plane of the pair as axis-plane in which the double-cone is generated.

Just as we had worked-out the hidden-views for the spherical-hypercone, like results occur also for the double-cone. If we take the hyperplane of the tetrahedron ABCO and rotate this tetrahedron around the plane of the face ABO as axis-plane, then the $\frac{1}{2}$-double-cone AB-$(\overline{CDE})$ will lie on one-side of the hyperplane of this tetrahedron and will be a visible-view in the graphic-figure except that portion made-up of the $\frac{1}{2}$-base in the interior of the $\frac{1}{2}$-circle $(\overline{CDE})$, and the other $\frac{1}{2}$-double-cone AB-$(\overline{EFC})$ will lie on the other-side of the hyperplane of this tetrahedron and will be a hidden-view.

To see this more clearly, observe that in the hyperplane of the blank-cone the $\frac{1}{2}$-arc CDE lies on one-side of the plane of the triangle ACE, and therefore, lies on one-side of the hyperplane of the tetrahedron ABCO; and the $\frac{1}{2}$-arc EFC lies on the other-side of the plane of the triangle ACE, and therefore, lies on the other-side of the hyperplane of the tetrahedron ABCO.

We can also consider the plane of symmetry ABO of the 2 tetrahedrons ABCO and ABEO as an axis-plane around which the rotation of the tetrahedron ABCO occurs; that is, the tetrahedron ABCE is composed of the 2 tetrahedrons ABCO and ABEO lying in the hyperplane of the tetrahedron ABCE, and a 180° counterclockwise-rotation around the face ABO will take this tetrahedron back again into the hyperplane of the tetrahedron ABCE, and therefore, this portion of the rotation will give us the visible-views in the graphic-figure of the $\frac{1}{2}$-double-cone AB-$(\overline{CDE})$ except that portion made-up of the $\frac{1}{2}$-base in the interior of the $\frac{1}{2}$-circle $(\overline{CDE})$; and the other $\frac{1}{2}$-rotation of the tetrahedron ABCO at ABEO will take the tetrahedron ABCO back again into the hyperplane of the tetrahedron ABCE at its initial-position at ABCO, and this $\frac{1}{2}$-portion of the double-cone will be a hidden-view in the graphic-figure.

We can generate the lateral-hypersurface of a double-cone AB-(CDE) in the following way: If we take the vertex-edge AB and take its extremity-points A and B with the point C of the circle (CED) whose interior is the base of the double-cone AB-(CDE), then the interior of the triangle ABC so formed will be 1 of the elements of the double-cone. Then if we take and rotate the point C around the point O in the plane of the circle (CDE), the traingle ABC with its vertex-edge fixed will rotate around the plane of the triangle ABO as axis-plane and its face will generate the lateral-hypersurface of the double-cone AB-(CDE), and its edges AC and BC will generate the lateral-surfaces of the 2 end-cones. The vertex-edge AB and its 2 extremity-points A and B do not belong to the lateral-hypersurface of the double-cone AB-(CDE), and the circle (CDE) does not belong to the lateral-hypersurface of the double-cone, for its interior is the base of the double-cone AB-(CDE). We could also describe the lateral-hypersurface of the double-cone as a hyperconical- hypersurface.

The visible and hidden-views of the elements of the double-cone AB-(CDE) are formed in the same-way that we had formed the visible and hidden-views of the $\frac{1}{2}$-double-cones that go to make-up the double-cone AB-(CDE). For example, the face ABD lies on one-side of the hyperplane of the tetrahedron ABCE and the face ABF lies on the other-side of the hyperplane of this tetrahedron, and in the traingle ADF, the face ADO lies on one-side of the line of symmetry AO and the face AFO lies on the other-side of the line of symmetry AO, this holds as well for the axis-plane of symmetry ABO lying in the hyperplane of the tetrahedron ABDF.

Another way would be to consider the edge AD lying on one-side of the plane of the triangle ACE and the edge BD lying on one-side of the plane of the triangle BCE, but 2 lines that intersect determine a plane and 2 planes that intersect determine a hyperplane, and therefore, the face ABD lies on one-side of the hyperplane of the tetrahedron ABCE formed of the planes of the 2 triangles ACE and BCE that intersect in the line of the edge CE.

We can also consider the double-cone AB-(CDE) as the hypercone B-A(CDE), in which case, the face ABC will generate its lateral-hypersurface, but will also include the lateral-surface of end-cone B-(CDE) generated by the edge BC of the triangle ABC.

58. SPHERES AND CIRCLES IN A HYPERSPHERE. TANGENT-HYPERPLANES. A HYPERSPHERE consists of the points at a given distance from a given point. The terms CENTER, RADIUS, CHORD, and DIAMETER are used as with circles and spheres.

Fig. 67 represents the graphic-form of a hypersphere which we shall denote by H. We graphically-construct H in a way analogous to the graphic-construction of a sphere in the 3-dimensional solid-geometry. In the Point-Geometry at its center O, we set-up a rectangular-system of 4 mutually-perpendicular lines which meet at O. The 4 mutually-perpendicular lines which meet at O will intersect H in 4 pairs of opposite-points (A,A'), (B,B'), (C,C'), and (D,D'). Each of these paired-points form the extremities of a diameter of H. The point O at the center of H separates the 4 mutually-perpendicular lines which meet at O into 4 pairs of opposite ½-lines of the rectangular-system, and intersect H in the above mentioned 4 pairs of opposite-points. The 4 axes of the rectangular-system in the graphic of H are constructed in much the same way that the axes of the hypercube was constructed: that is, we will have foreshortening of the diameters BB' and DD' lying on

2 of the great -spheres of H in this figure are represented as having the appearance of ellipsoids(not 3 as text) The great-spheres S_{ABS} &SACD in the DOP-graphic are represented like our ordinary sphere in 3 space graphic. Other than SACD, each sphere will have the extremity points of 2 of their major axis of ellipses represented as great circles lying on SACD as well..that is for each of SABC SBCD and SABD.... (correction)AF

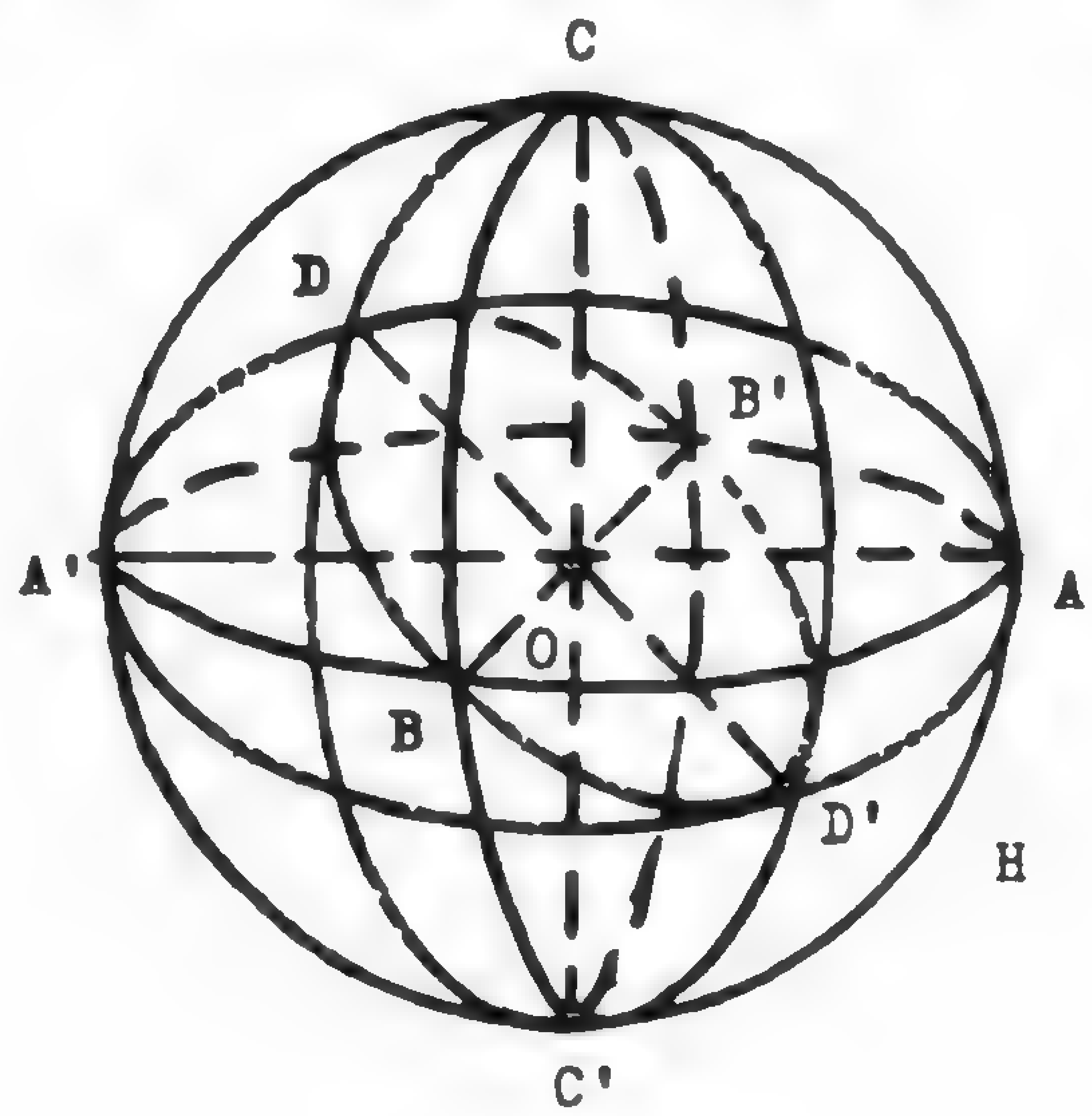


Fig. 67.

the oblique-axes which contain them respectively, with BB' greatly-foreshortened, and DD' moderately-foreshortened; and the diameters AA' and CC' lying on the axes which contain them respectively, and which are at 'true' right-angles to each other in the plane of the paper, will be non-foreshortened. In each of the 6 planes about the point O in the graphic except the plane of the circle (CAC'), we construct an ellipse representing a great-circle in H—we shall go into the details of the graphic-construction of these ellipses in another part of this section.

The 4 great-spheres of H in the graphic-figure will be denoted by S_{ABC}, S_{ABD}, S_{ACD}, and S_{BCD}. For example, S_{ABC} means that this great-sphere is determined by the 3 points A, B, and C not coplanar with the center O, and that any 1 of the 3 points A, B, and C will be the pole of the plane of the great-circle containing the other 2 points. We can also use the above notation-form for a great-sphere in the case when 1 of the 3 points A, B, and C will not be the pole of the plane of the great-circle containing the other 2 points. There are 3 possibilities here, but we shall use the notation-form for a great-sphere in the case when 1 of the 3 points A, B, and C is the pole of the plane of the great-circle containing the other 2 points; and, which will greatly-simplify our study of the hypersphere.

We shall consider 1st the graphic-form of the black-sphere S_{ABC} in Fig. 67. In the graphic-construction of the black-sphere, 2 of its great-circles will be represented as ellipses. Now, if we take into consideration the definition of a sphere, then the EXTENT of S_{ABC} in its graphic-form cannot exceed any 1 of the diameters of the circle (CAC') lying in the plane of the paper, and therefore, all other great-circles of the black-sphere represented as ellipses in the graphic must lie within the interior of (CAC') except that the extremity-points of the major-axis on each of these ellipses will lie on (CAC'). The extremities of BB' lie in the interior of (CAC')

Now in the graphic-construction of the hypersphere H, we use an analogous-method corresponding to the way in which we had graphically-constructed the black-sphere S_{ABC}. We graphically-construct the red-sphere S_{ACD} in the same way that we had graphically-constructed the black-sphere S_{ABC}; that is, since the circle (CAC') lies in the plane of the paper, which is also a plane in the hyperplane of the red-sphere S_{ACD}, we can graphically-construct S_{ACD} in the plane of the circle (CAC'). DD' will lie in the interior of the circle (CAC'), and in the graphic-form of S_{ACD}, the extent of S_{ACD} cannot exceed any 1 of the diameters of (CAC'), and therefore, all other great-circles of the red-sphere S_{ACD} represented as ellipses in the graphic must lie within the interior of (CAC') except that the extremity-points of the major-axis on each of these ellipses will lie on (CAC').

We graphically-construct the 3 great-spheres S_{ABC}, S_{ABD}, and S_{BCD}, which in the graphic are represented as ellipsoids lying entirely in the interior of the graphic-form of the red-sphere S_{ACD} (in its hyperplane) except that each of the 3 great-spheres represented as ellipsoids will have 1 of their ellipses lying on the graphic-form of the red-sphere S_{ACD}. The reason for this is analogous to that given for the graphic-construction of the black-sphere S_{ABC}; that is, assuming the definition of a hypersphere, then the EXTENT of H in the graphic cannot exceed any 1 of the diameters of the red-sphere S_{ACD}. Since the axes DD' and BB' of the red-sphere S_{ACD} and the black-sphere S_{ABC} respectively, intersect at O, then the plane determined by these 2 axes will intersect H in the great-circle (DBD') represented in the graphic as an ellipse. The great-circle (DBD') intersects the great-sphere S_{ACD} in the 2 points D and D', which are the extremity-points of a diameter of (DBD'); whereas, the 2 points B and B' lie on opposite-sides of the hyperplane of the ~~red-great~~-sphere S_{ACD}. But just as the 2 points B and B' lie in the interior of (CAC') in the plane of the paper, so too, the great-circle (DBD') represented as an ellipse in the graphic will lie in the interior of the graphic of the red-great-sphere S_{ACD} in its hyperplane. The 2 great-circles (CAC') and (DBD') in the graphic can not intersect, for the planes of these great-circles are absolutely-perpendicular, and therefore (CAC') and (DBD') cannot intersect in the extremities of any diameter of H.

The ellipses representing great-circles in the graphic-form of H are relatively easy to construct. For example, consider the ellipse CDC'D' and the points C, D, C', and D' lying on it, then if we pass a line perpendicular to its major-axis CC' at its center at O in the plane of the paper, we can form the minor-axis of this ellipse and graphically-construct the ellipse CDC'D'; and use this method of graphic-construction for the other ellipses in the graphic of H. It should be noted that the ellipse DBD'B' representing the great-circle (DBD') has as its major-axis DD', which has its 2 extremity-points D and D' lying on the graphic of the red-sphere S_{ACD}, and therefore must be a major-axis of this ellipse; whereas, the extremity-points B and B' of BB' do not lie on the graphic of the red-sphere S_{ACD}, and therefore cannot be a major-axis of this ellipse.

Many other details of the graphic-form of H can be derived by considering the Point-Geometry at its center and the geometry of the hypersphere itself, and then deriving the corresponding graphic-relationships.

The SOP and DOP-graphics of the hypersphere H will be different only in the degree of their scale-distortion-factors and some differences in the overlapping of geometric-points in the graphic, otherwise much alike. In the DOP-graphic of a hypersphere H, we have the following graphic-characteristics: 2 of the 4 great-spheres will be represented as ellipsoids, and 2 will be represented like our ordinary-sphere; the red-great-sphere S_{ACD} will not be a true-sphere due to its scale-distortion-factor of 2 (double oblique-axes and scale-distortion on each).

In the SOP-graphic we would have the following graphic-characteristics: 1 of the 4 great-spheres in H will be a true-great-sphere without any scale-distortion whatsoever, and this true-great-sphere will be the red-great-sphere S_{ACD}; and the other 3 great-spheres will be somewhat like our 3-space black-sphere having a single scale-distortion, and therefore will not have the appearance of ellipsoids in the 3-space flat-hypersurface (Euclidean-hyperplane) upon which the hypersphere H is graphically-constructed, for the observer will lie in 4-space outside of the hyperplane of the great-sphere S_{ACD}. In the SOP-graphic, then, we would have 3 true-circles, and 3 ellipses representing 3 great-circles; whereas, in the DOP-graphic, we will have 1 true-circle, and 5 ellipses representing great-circles in the hypersphere H.

Just as the great-circle (CAC') was a visible-view in the 3-space graphic of the black-sphere, so too, in the 4-space graphic of H, the red-sphere S_{ACD} will be completely

visible. We shall prove that the red-sphere S_{ACD} is a visible-view in the graphic when we make a study of the rotation of the rotation of a hypersphere in hyperspace.

Theorem 1. Any hyperplane-section of a hypersphere is a sphere having for center the projection of the center of the hypersphere upon the hyperplane (Art. 13ª Th. 2).

When the hyperplane passes through the center of the hypersphere the section is a GREAT-SPHERE. Other spheres of the hypersphere are SMALL-SPHERES.

Theorem 2. 4 non-coplanar points of a hypersphere determine a sphere of the hypersphere, and 3 points non-coplanar with the center of the hypersphere determine a great-sphere.

Theorem 3. Any plane having more than 1 point in a hypersphere intersects the hypersphere in a circle having for center the projection of the center of the hypersphere upon the plane.

Theorem 4. 3 points of a hypersphere determine a circle, and 2 points not collinear with the center of the hypersphere determine a great-circle.

Theorem 5. 2 great-circles on the same great-sphere intersect, and 2 great-circles which intersect lie on 1 great-sphere.

A great-circle and a great-sphere always intersect, intersecting in the extremities of a diameter, and 2 great-spheres intersect in a great-circle (Art. 4, Ths. 1 and 2).

As an example of Th. 5, the great-circle (CDC') intersects the great-sphere S_{ABC} in the extremity-points C and C' of the diameter CC', and the 2 great-spheres S_{ABC} and S_{BCD} intersect in the great-circle (CBC').

Theorem 6. 2 great-spheres of a hypersphere bisect each other.

DISTANCE IN A HYPERSPHERE between 2 points not the extremities of a diameter is always measured on the arc less than 180° of the great-circle containing them. The distance between the extremities of any diameter is 180°.

Theorem 7. All the circles of a hypersphere which pass through a given point are perpendicular, that is, their tangents are perpendicular, to the radius of the hypersphere at this point.

A hyperplane perpendicular to a radius of a hypersphere at its extremity is TANGENT TO THE HYPERSPHERE.

59. HYPERLUNES AND SPHERICAL-DIHEDRAL-ANGLES. SPHERICAL-TETRAHEDRONS. A great-circle of a sphere dividing the rest of the sphere into 2 hemispheres may be called the EDGE of either of these hemispheres. 2 hemispheres of great-spheres having a common-edge will enclose a portion of the hypersphere of definite-volume, and we shall call that portion of a hypersphere bounded by 2 hemispheres of great-spheres a HYPERLUNE. The hyperlune is the analogue to the lune in the spherical-geometry of a sphere, a lune being defined as that portion of a sphere bounded by 2 intersecting arcs of great-circles.

Along the edge we have a SPHERICAL-DIHEDRAL-ANGLE, which consist of a restricted-portion of the edge and restricted-portions of the hemispheres. The restricted-portion of the edge lies between the extremities of a diameter which divides the edge into 2 parts on opposite-sides of each other, and we shall call these opposite-parts of the edge the DOUBLE-EDGE OF 2 OPPOSITE SPHERICAL-DIHEDRAL-ANGLES, with 1 of each of the edges of the double-edge belonging to 1 of the opposite spherical-dihedral-angles.

The edge of a spherical-dihedral-angle has on each face a pole, and the arcs of great-circles drawn through these points from any point of its edge determine a spherical-angle by which the spherical-dihedral-angle can be measured, just as the dihedral-angle formed by 2 ½-planes is measured by its plane-angles.

The spherical-dihedral-angle is itself measured by the distance between the poles of the edge, so that the distance can be considered as a measure of the spherical-dihedral-angle, and if we take corresponding units, as a measure of the volume enclosed by the 2 hemispheres, we shall have the volume of the hyperlune.

The edge and poles of a spherical-dihedral-angle determine 2 lunes each of which has a spherical-angle of 90°, and the interiors of which are its faces. 2 intersecting hemispheres bounding a hyperlune determine 2 lunes of a spherical-dihedral-angle.

The tangent $\frac{1}{2}$-planes which have a common-edge tangent to the edge of a spherical-dihedral-angle form an ordinary dihedral-angle whose measure can be taken as the measure of the former.

Notation: S_{BC}^{A} denotes a hemisphere of S_{ABC}, where A is the pole of the plane BCO containing the edge of this hemisphere. The hemisphere opposite to S_{BC}^{A} will be denoted by $S_{BC}^{A'}$. Thus, we see that by raising a subscript-letter of S_{ABC}, we can determine 1 of its hemispheres. By raising a subscript-letter and priming it we shall have another set of hemispheres. When 2 hemispheres in the notation-form have the same subscripts and same superscripts with 1 of the superscripts primed, then the 2 hemispheres are said to be on opposite-sides of each other. For example, $S_{ABC} = S_{BC}^{A} + S_{BC}^{A'}$, each have different poles but the same edge, that is, the points A and A' are opposite to each other and are the extremities of a diameter AA'. The 2 hemispheres S_{BC}^{A} and S_{BC}^{D} lie on different great-spheres in H, but have the same edge. The notation for the hemispheres of S_{ABD}, S_{ACD}, and S_{BCD} are denoted in the same manner as explained above for S_{ABC}, but with different subscript and superscript-letters.

In Fig. 67, the 2 intersecting hemispheres S_{BC}^{A} and S_{BC}^{D} determine a hyperplune in the hypersphere H. We shall desiginate the spherical-dihedral-angle of the hyperlune by D-CBC'-A. The spherical-dihedral-angle D-CBC'-A is formed by the intersection of the 2 lunes CBC'DC and CBC'AC. The faces of D-CBC'-A are the interiors of the lunes CBC'DC and CBC'AC each of which lies on 1 of the hemispheres of the hyperlune, and the intersecting edge CBC' of these 2 lunes is the edge of D-CBC'-A. The edge CBC' and the 2 poles D and A of D-CBC'-A determine the 2 lunes CBC'DC and CBC'AC, each of which has a spherical-angle of 90°. The spherical-angle of D-CBC'-A is measured by the distance between the 2 poles A and D of the edge CBC', and this distance is the arc AD of the great-circle (ABA').

The spherical-dihedral-angle D-CBC'-A can also be determined in the following way: Take the point B on the edge CBC' as the pole of the great-sphere S_{ACD} (see Art. 60), then the 2 hemispheres S_{BC}^{A} and S_{BC}^{D} will intersect S_{ACD} in the arcs CAC' and CDC' respectively, and the great-circle (ABA') of S_{ACD} in the points A and D, respectively; then if we take the point B with the points A, C, and D, we shall have 3 intersecting arcs at B which intersect S_{ACD} in the points A, C, and D. The spherical-dihedral-angle D-CBC'-A formed of these 3 arcs will have its spherical-angle at the point B measured on the great-sphere S_{ACD}. In the hyperplane of S_{ACD}, then, C will be the pole of the plane of the great-circle (ABA'), and in the plane of this great-circle, we shall have a plane-angle AOD at O which can be taken as the measure of the spherical-angle of D-CBC'-A; that is, on S_{ACD} we take 2 intersecting arcs CDC' and CAC' forming a spherical-angle at C, with C being its vertex, and the spherical-angle of these 2 intersecting arcs is measured by the arc AD of the great-circle (ABA') having the vertex C of the spherical-angle of these intersecting arcs as pole and is included between the sides of the spherical-angle formed of these 2 intersecting arcs.

The spherical-dihedral-angles and spherical-angles correspond to the Point-Geometry at the center of the hypersphere. Just as we measured the plane-angle of a hyperplane-angle, we can take the intersections of the hyperplanes of the 2 hemispheres of a hyperlune and form a hyperplane-angle, and the plane-angle of the hyperplane-angle will have at O (the center of H) the same measure as the spherical-angle of the spherical-dihedral-angle of the hyperlune (see Art.27, Ths. 2 and 4).

A $\frac{1}{2}$-hypersphere is a hyperlune whose spherical-dihedral-angle is 180°, and a hypersphere is a hyperlune whose spherical-dihedral-angle is 360°.

Theorem 1. A spherical-dihedral-angle has the same measure at all points of its edge. (see Art. 27, Th. 4)

Theorem 2. The volume enclosed by the hemispheres of a spherical-dihedral-angle is to the volume of the hypersphere as the dihedral-angle is to 4 right-angles.

The formula for this is determined by considering the volume of a hyperlune of 1°, which is $2\pi^2r^3/360$ (π = Pi), or $\pi^2r^3/180$, and therefore the volume of a hyperlune of spherical-dihedral-angle θ (θ = Theta) is equal to $\pi^2r^3\theta/180$. We shall denote this volume by S_3, we then have

$$S_3 = \pi^2r^3\theta/180 .$$

When $\theta = 360^{\circ}$, we have the volume of a hypersphere, where $S_3 = 2\pi^2 r^3$. Th. 2 can now be put in an equation-form as follows:

$$\frac{S_3}{2\pi^2 r^3} = \frac{\theta}{360} .$$

We define SPHERICAL-TRIHEDRAL-ANGLE and SPHERICAL-TETRAHEDRONS in the hyperspherical-geometry in an analogous way that we had defined spherical-angle and spherical-triangles in the spherical-geometry. We shall suppose that the sides of a spherical-triangle and the edges of a spherical-tetrahedron are less than 180°. The 4 great-spheres which contain the faces of a spherical-tetrahedron determine a set of 16 spherical-tetrahedrons, 8 pairs, the 2 spherical-tetrahedrons of a pair being congruent (Art. 49, Th. 2). The $\frac{1}{2}$-hypersphere which lies on one-side of any 1 of the 4 great-spheres (on one-side of the hyperplane of the great-sphere) contains the interiors of 8 spherical-tetrahedrons, 1 from each pair.

A spherical-tetrahedron has 6 edges, each lying in the edge of a spherical-dihedral-angle whose interior contains the interior of the spherical-tetrahedron. The interiors of 1 of these spherical-dihedral-angles contains also the interiors of 3 of the 15 spherical-tetrahedrons associated with the given spherical-tetrahedron as explained above.

The 4 great-spheres S_{ABC}, S_{ABD}, S_{ACD}, and S_{BCD} determine a set of 16 spherical-tetrahedrons in the hypersphere H of Fig. 67, which we shall call a NETWORK OF SPHERICAL-TETRAHEDRONS ON THE HYPERSPHERE H. In fact, the network of these spherical-tetrahedrons are REGULAR-SPHERICAL-TETRAHEDRONS, defined as having all their sides equal to 90°; spherical-face-angles, spherical-dihedral-angles, and spherical-angles are all equal to 90°. The network of these 16 spherical-tetrahedrons completely fill-up the hypersphere H (its hypersurface, that is, its spherical-hypersurface). For example, the spherical-tetrahedron ABCD has all its sides and angles equal to 90°, and ABCD has 4 vertices and 6 edges, the edge CB lies in the spherical-dihedral-angle D-CBC'-A and which contains the interior of ABCD, and the spherical-angle of D-CBC'-A is 90°. Other details and developments can be immediately read-out from the graphic-figure of H, and so we will not make a detailed-study of these spherical-tetrahedrons in this small treatise.

60. POLES AND POLAR-CIRCLES. The diameter of a hypersphere perpendicular to the hyperplane of any sphere of the hypersphere is called the AXIS OF THE SPHERE, and the extremities of the axis are called the poles of the sphere.

Example. In Fig. 67, the diameter DD' is perpendicular to the hyperplane of the great-sphere S_{ABC} at the point O, DD' is its axis, and the extremities D and D' of DD' are its poles.

Theorem 1. Each pole of a sphere of a hypersphere is equidistant from all the points of the sphere.

Example. In Fig. 67, the point D of H is a pole of S_{ABC}. The distances DA, DB, DC, DA', and DB' are all equal. The distance between D and any point on S_{ABC} will be equal to a quadrant.

The plane through the center of a hypersphere absolutely-perpendicular to the plane of any circle of the hypersphere is called the AXIS-PLANE OF THE CIRCLE, and the great-circle in which this plane intersects the hypersphere is the POLAR-CIRCLE OF THE GIVEN ~~GREAT-~~ CIRCLE.

Theorem 2. Each point of the polar-circle of a circle of a hypersphere is equidistant from all the points of the given circle.

Theorem 3. A great-circle of a hypersphere is itself polar to the great-circle which is its polar, and the distance between any 2 points, 1 in each of 2 polar great-circles, is a quadrant.

Example. In Fig. 67, the 2 great-circles (CDC') and (ABA') are polar great-circles lying in H.

Theorem 4. A great-sphere contains all the points at a quadrant's distance from either of its poles, and each of 2 polar great-circles contains all the points at a quadrant's distance from each other.

Theorem 5. Great-circles which pass through the poles of a sphere are perpendicular to the sphere, and any great-circle perpendicular to a sphere passes through the poles of the sphere.

If a point moves a given distance along an arc of a great-circle, its polar great-sphere will rotate around the polar great-circle and generate a spherical-dihedral-angle whose measure is this same distance.

Example. In Fig. 67, if the point B moves along an arc of a great-circle (ABA') to the position of the point A on the great-circle, then the distance generated will be the quadrant BA. Now if we take the polar great-sphere S_{ACD} of B, then the polar great-circle of (ABA') will be (CDC'), and S_{ACD} will rotate around (CDC') and generate a spherical-dihedral-angle; and, if we take CDC' as 'edge' of the generated spherical-dihedral-angle, then we shall have the spherical-dihedral-angle A-CDC'-B, and therefore, the spherical-angle AB as the measure of A-CDC'-B, and this distance is the same distance BA that the point B generated on an arc of (ABA').

If a great-circle rotates on a great-sphere through a given spherical-angle around 1 of its points (and the opposite-point), its polar great-circle, lying on the polar great-sphere of the given point and passing through the pole of the given great-sphere will rotate around the latter through the same angle.

Example. In Fig. 67, if a great-circle (CBC') rotates on a great-sphere S_{ABC} through a spherical-angle BA around 1 of its points, say C (and its opposite-point C'), its polar great-circle (ADA'), lying on the polar great-sphere S_{ABD} of C and passing through the pole D of S_{ABC}, will rotate around D through the same spherical-angle BA. The 2 points C and D determine the great-circle (CDC'), and the rotation takes place in the hypersphere H around the plane of (CDC') (see Art. 47).

Polar spherical-tetrahedrons correspond to polar spherical-triangles in the spherical-geometry. The properties of polar spherical-tetrahedrons is a specialized-topic of the hyperspherical-geometry, and somewhat involved, though an interesting study in itself.

To show that the geometry of the hypersphere is an independent 3-dimensional geometry, see Manning's Geometry of Four Dimensions—Art. 122, pp. 212-215.

61. POINT-GEOMETRY THE SAME AS THE GEOMETRY OF THE HYPERSPHERE.

Theorem. The Point-Geometry at the center of a hypersphere is the same as the geometry of the hyperpshere. (Fig. 67.)

Given: The Point-Geometry at the center O of a hypersphere H.

To Prove: The Point-Geometry at O is the same as the geometry of H.

Proof: In the hypersphere H, points, great-circles, and great-spheres are its intersections with ½-lines drawn from the center at O and with planes and hyperplanes through its center at O, and the distances (arcs of circles) and angles in the hypersphere H are the same as the corresponding angles at O. Therefore the Point-Geometry at O is the same as the geometry of H. (Q.E.D)

In particular, to a great-sphere of the hypersphere and its poles correspond at the center a hyperplane and its perpendicular ½-lines; to 2 polar great-circles correspond 2 absolutely-perpendicular planes; and 2 simply-perpendicular planes correspond to 2 great-circles intersecting at right-angles.

The theorems of the Point-Geometry can, then, be stated as the theorems of the geometry of the hypersphere.

62. LOGIC-DIAGRAM GRIDS ASSOCIATED WITH PORTIONS OF THE GRAPHIC-FIGURE OF A HYPERSPHERE. HEXADEKANTS AND THE PRINCIPLE-HEXADEKANT.

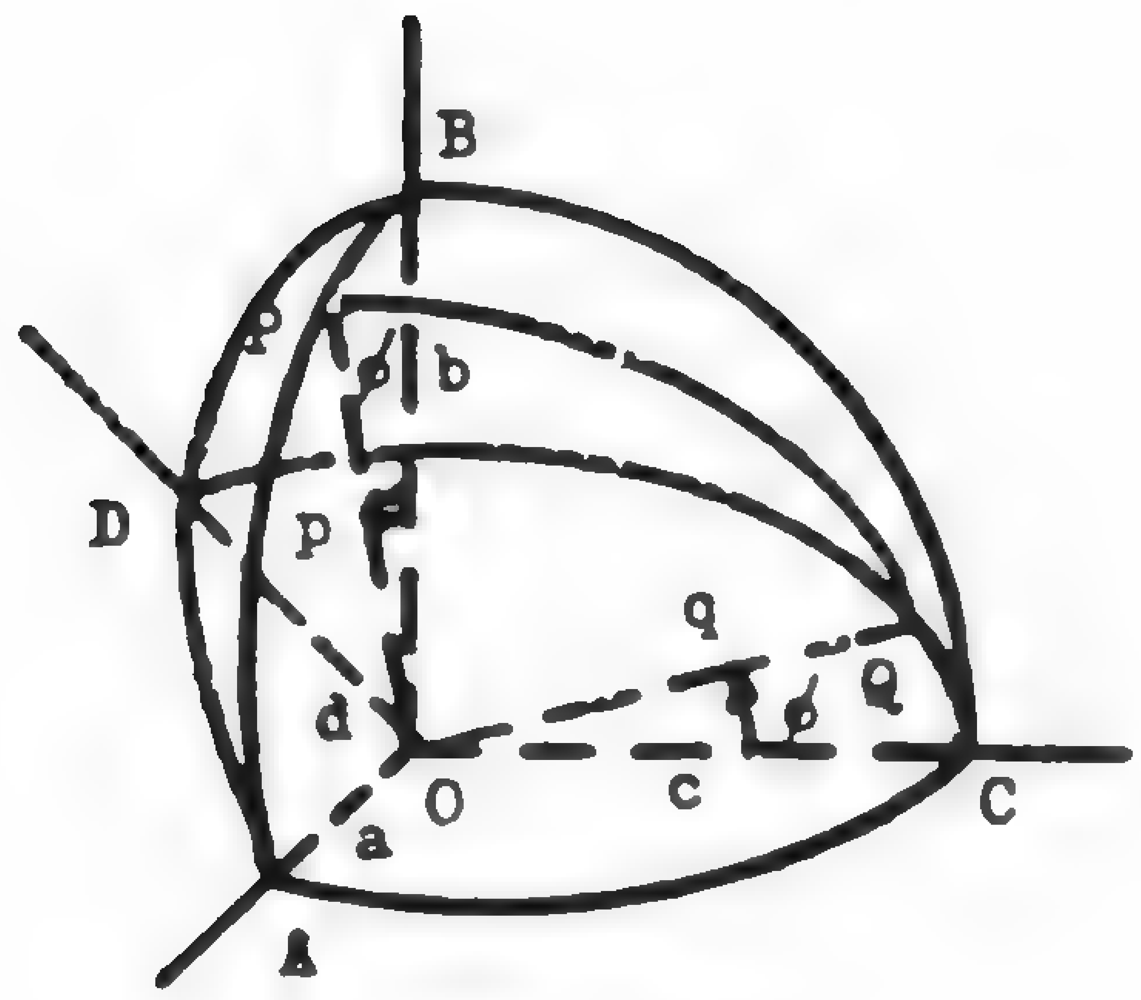
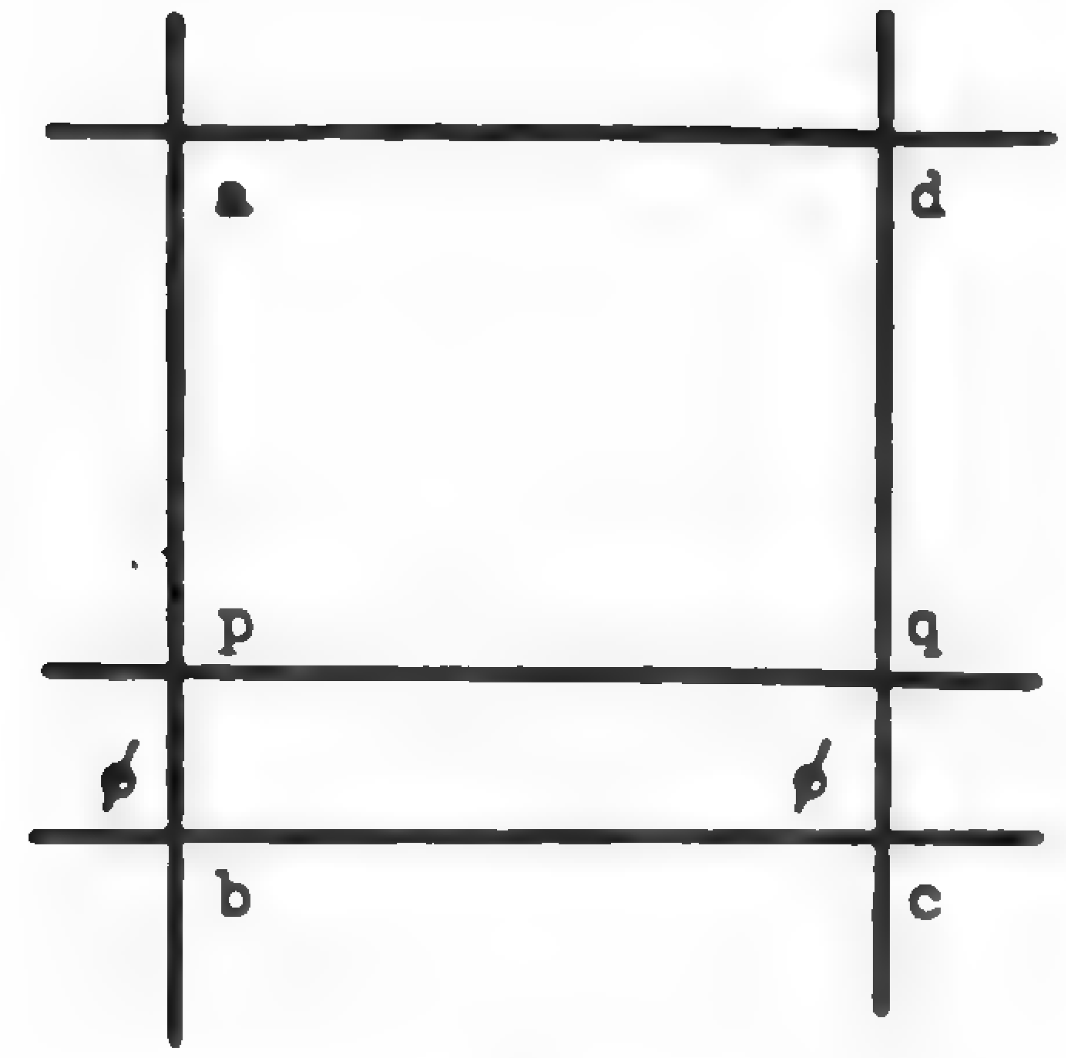


Fig. 68.



Logic-Diag. (GRID) 8.

In the Point-Geometry at the center of a hypersphere, 4 mutually-perpendicular hyperplanes intersecting at its center will divide the hypersphere about its center into 16 regions called HEXADEKANTS. Any 3 out of 4 of these mutually-perpendicular hyperplanes determine a line through the center of the hypersphere, altogether, we shall have 4 mutually-perpendicular lines of a rectangular-system. The 4 mutually-perpendicular lines which meet at the center of the hypersphere are divided by the point at the center of the hypersphere into 4 pairs of opposite $\frac{1}{2}$-lines of a rectangular-system. If we take any 4 $\frac{1}{2}$-lines out of the 8 $\frac{1}{2}$-lines of the rectangular-system thus determined except pairs of opposite $\frac{1}{2}$-lines, then any 1 of a set of the 4 $\frac{1}{2}$-lines thus determined will lie in 1 of the 16 regions of the hyperspace about the center of the hypersphere. The hexadekant containing the 4 mutually-perpendicular $\frac{1}{2}$-lines lying in the positive-portion of a rectangular-system will be called the PRINCIPAL-HEXADEKANT.

We then form a LOGIC-DIAGRAM GRID representing the relationships between the Point-Geometry at the center of a hypersphere and the geometry of the hypersphere, and associate it with that portion of the hypersphere lying in the region of the principal-hexadekant. The logic-grid will then be in 1-1 correspondence with that portion of the graphic-figure of a hypersphere lying in the principal-hexadekant.

The GRID of Logic-diag. 8 is formed by the 2 pairs of absolutely-perpendicular planes (bc, ad) and (ba, cd). We can also call the grid thus formed, the PRINCIPAL-GRID, since this grid corresponds to that portion of the hypersphere lying in the principal-hexadekant of a rectangular-system.

Fig. 68 represents the graphic for that portion of a hypersphere H lying in the principal-hexadekant of a rectangular-system and which corresponds to the principal-grid of the logic-diagram.

However, a subtle-DIFFERENCE show be noted between the logic-grid and the corresponding graphic-portion of the hypersphere H lying in the principal-hexadekant. The $\frac{1}{2}$-lines represented as points in the logic-grid will correspond to $\frac{1}{2}$-lines in the graphic and will actually be represented as $\frac{1}{2}$-lines in the graphic. We shall consider only that portion of the $\frac{1}{2}$-lines in the graphic lying between the point O and their intersections with the hypersphere H.

The logic-grid enables us to determine directly, in the graphic, those points lying on that portion of the hypersphere H which lies in the principal-hexadekant. We take any point in the logic-grid representing a $\frac{1}{2}$-line in the Point-Geometry at the center of H and then form the corresponding $\frac{1}{2}$-line in the graphic. The point at which the $\frac{1}{2}$-line intersects the hypersphere H, will be a point lying on H. Any 2 points in the logic-grid representing $\frac{1}{2}$-lines in the Point-Geometry at the center of H will correspond to 2 $\frac{1}{2}$-lines in the graphic, and which will lie in the plane of a great-circle.

After we have determined the points A,B,... lying on H in the graphic, we can then substitue these points (on the $\frac{1}{2}$-lines on which they lie) for the corresponding points in

in the logic-grid, that is, we can let the points A,B,... in the graphic correspond to the points a,b,... in the logic-grid, but with the understanding that the $\frac{1}{2}$-lines a,b,... represented as points in the logic-grid are now interpreted as points at a given distance from O (which lie on H); thus, we see that the geometrical-relationships between the graphic and logic-grid are in 1-1 correspondence.

We shall give a few examples illustrating the relationships between the logic-grid and graphic as follows: the quadrant-plane ba in the logic-grid determined by the 2 points b and a representing $\frac{1}{2}$-lines in the Point-Geometry at the center of H will correspond to the quadrant-plane ba in the graphic determined by the 2 $\frac{1}{2}$-lines b and a; the length of bp in the logic-grid will correspond to the arc BP in the graphic, that is, the arc BP lying on the great-circle determined by the 2 points B and P, lies also in the plane of this great-circle determined by the 2 $\frac{1}{2}$-lines b and p; the angle ϕ in the logic-grid corresponds to the angle ϕ in the Point-Geometry at the center of H.in the graphic.

In the next section we will go into more detail in a study of the relationships between the logic-grid and graphic.

63. CLIFFORD'S PARALLELS. PARALLEL GREAT-CIRCLES CORRESPONDING TO THE ISOCLINE-PLANES OF THE POINT-GEOMETRY. Since the theorems of the Point-Geometry can be stated as theorems of the geometry of the hypersphere, we shall mention only some of the more important results (Arts. 51, 52, 53, and some other ths. of the Point-Geometry), and make use of the logic-diagram grid and corresponding graphic. (Fig. 69, Logic-diag. (GRID) 9.)

Notation: Given a hypersphere H, we will write GREAT-CIRCLE AB to denote the great-circle determined by the 2 points A and B lying on H, and we will write DISTANCE AB to denote the distance determined by the 2 points A and B on H.

Any 2 great-circles have a pair of common-perpendicular great-circles, 2 polar great-circles which intersect them at right-angles.

Example. The great-circles BC and PQ have a pair of common-perpendicular great-circles BA and CD.

When 2 great-circles cut-out equal arcs on a polar pair of common-perpendicular great-circles, they have an infinite-number of common-perpendicular great-circles, on all of which they cut-out the same arc. Conversely, if 2 great-circles have more than 2 common-perpendicular great-circles, the arcs not greater than a quadrant which they cut-out on any 1 of them and on its polar great-circle are equal.

Example. The 2 great-circles BC and PQ cut-out equal arcs BP and CQ on their polar pair of common-perpendicular great-circles BA and CD respectively, and therefore have an infinite-number of common-perpendicular great-circles, on all of which they cut-out the same arc. The great-circles BC and PQ cut-out an arc RU on the great-circle RS, and since RU = BP, where BP = CQ, then the great-circle RS is 1 of the common-perpendicular great-circles of the 2 great-circles BC and PQ. In the Point-Geometry at the center of H, the plane-angles corresponding to the arcs BP, RU, and CQ are all equal, that is, they are all equal to ϕ.

There are 2 DISTANCES BETWEEN 2 GREAT-CIRCLES, the distance not greater than a quadrant measured along a polar pair of common-perpendicular great-circles. When the distances are equal the given circles are PARALLEL in the sense used by Clifford. Parallel great-circles correspond to the isocline-planes of the Point-Geometry.

Example. If we take the 2 great-circles BC and PQ, then their polar pair of common-perpendicular great-circles BA and CD will measure the 2 distances BP and CQ respectively, on these 2 great-circles. These 2 distances are less than a quadrant, and since BP = CQ, the 2 great-circles BC and PQ are parallel.

There are 2 senses in which great-circles can be parallel, and 2 great-circles perpendicular to both of 2 parallel great-circles (which are not polar) are themselves parallel in the opposite-sense. Through any point not a point of a given great-circle nor a point of its polar great-circle pass 2 great-circles parallel in the 2 senses to the given circle and to its polar. 2 great-circles parallel to a given great-circle in the same sense are parallel to each other in this sense also; and the set of all the great-circles parallel to a given great-circle in a given sense completely fills the

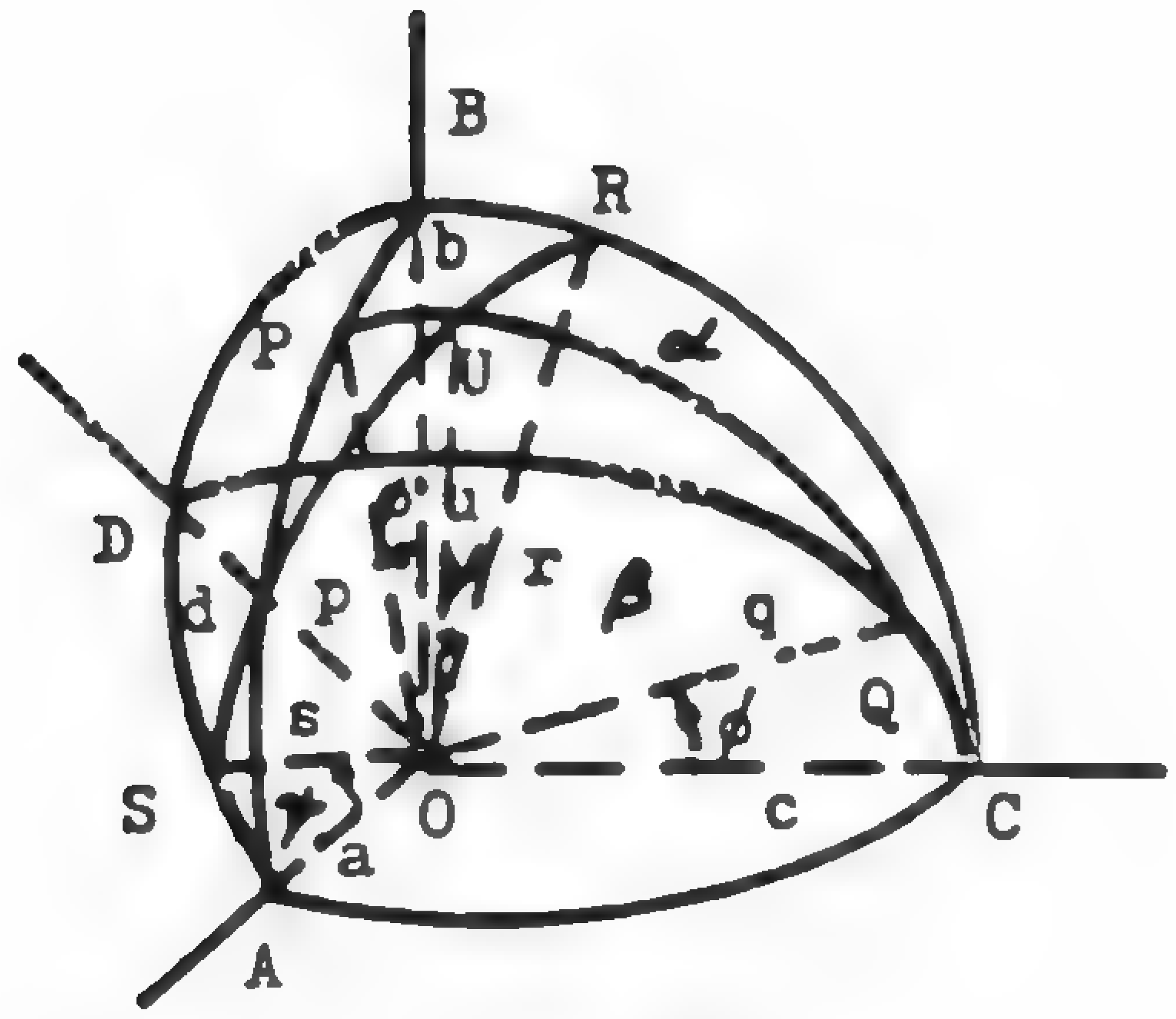
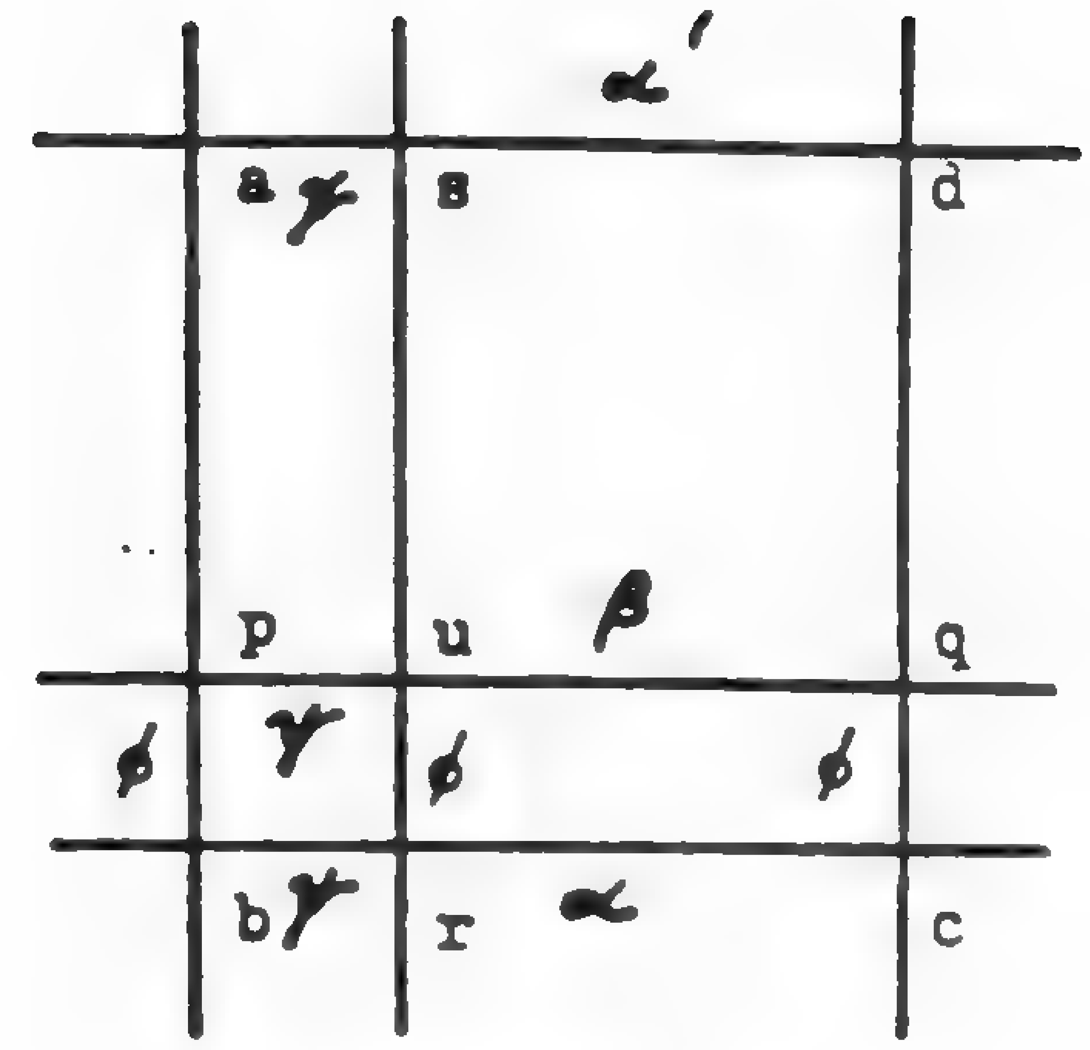


Fig. 69.



Logic-Diagram (GRID) 9.

hypersphere, 1 and only 1 circle passing through each point. (Use Fig. 60, Logic-diag. 3; complete the graphic-construction of that portion of H lying in the principal-hexadekant and in a portion of another hexadekant, and 1 logic-grid with a portion of another logic-grid.)

We conclude this section by stating some of the properties of a hypersphere H lying in the principal-hexadekant: That portion of a hypersphere H lying in the principal-hexadekant will be 1/16 of H; the volume enclosed will be 1/16 of the volume of H, and will be that of the volume of the spherical-tetrahedron ABCD; since the edges of ABCD are all equal to a quadrant (arcs of 90°), then ABCD is a SELF-POLAR spherical-tetrahedron, for any 1 of its faces will have a vertex-point opposite to it as POLE; at any vertex of ABCD, we have a trirectangular spherical-trihedral-angle; all of its face-angles will be right spherical -angles; and, any edge of ABCD will belong to an edge of a right spherical-dihedral-angle.

64. ROTATION OF THE HYPERSPHERE. Rotation of the hypersphere on itself is the same as the rotation of the Point-Geometry at its center. In any simple-rotation a certain great-circle remains fixed in all of its points; while its polar great-circle, the circle of rotation, rotates or slides on itself (Art. 47).

A DOUBLE-ROTATION is a combination of 2 simple-rotations around 2 polar great-circles.

Theorem. If in the hyperplane of a sphere of revolution we pass a plane through its axis and rotate around this plane that portion of the sphere which lies on one-side of it, we shall have all of a hypersphere except that portion which makes-up the section of the sphere by the plane. (Fig. 70.)

In the hyperplane of the sphere of revolution S_{ACD}, if we take the plane of the great-circle (CDC') passing through its axis CC' and rotate around the plane of this great-circle the $\frac{1}{2}$-sphere $S^{A'}_{ACD}$, we shall have all of the hypersphere H except that portion which makes-up the section of the sphere S_{ACD} by the plane of the circle (CDC'). In other words, if we take the plane of the circle (CDC') as axis-plane and rotate around this axis-plane, we shall have all of H except that portion of the circle (CDC') and its interior.

If we rotate the $\frac{1}{2}$-sphere $S^{A'}_{ACD}$ around the great-circle (CDC') as the axis of rotation, then (CDC') will remain fixed in all of its points; and, the pole A' of $S^{A'}_{ACD}$, being a point of the polar great-circle (ABA'), will rotate around (CDC') on its polar great-circle (ABA'), that is, the path of A' will move along the polar great-circle (ABA').

A surface of double-revolution will not be taken up in this short-treatise. More advanced-knowledge of the Point-Geometry is required, which involves considerable detail. However, for those interested in this study as such, will find that a surface of double-revolution has the appearance of an anchor-ring. For further details, see Manning's Geometry of Four Dimensions, pp. 219-220.

Still more [illegible]: A sphere has 4π (4 pi) steradians as a [illegible] of a solid-angle of 360°, but on a hypersphere, a complete [illegible] is $2\pi^2$ (2 pi square steradians).

Here's one that will cause mathematicians to go blewy: Consider the visual-hypersphere in Fig. 10, p-84 of the B-book, how does one measure the solid-spherical-angle of the [illegible] between ABCD? Still more madness... the hypersurface of a [illegible] is a 1-sided hypersurface in hyperspace, the analogue to the 1-sided surface of a [illegible] in 3-space.

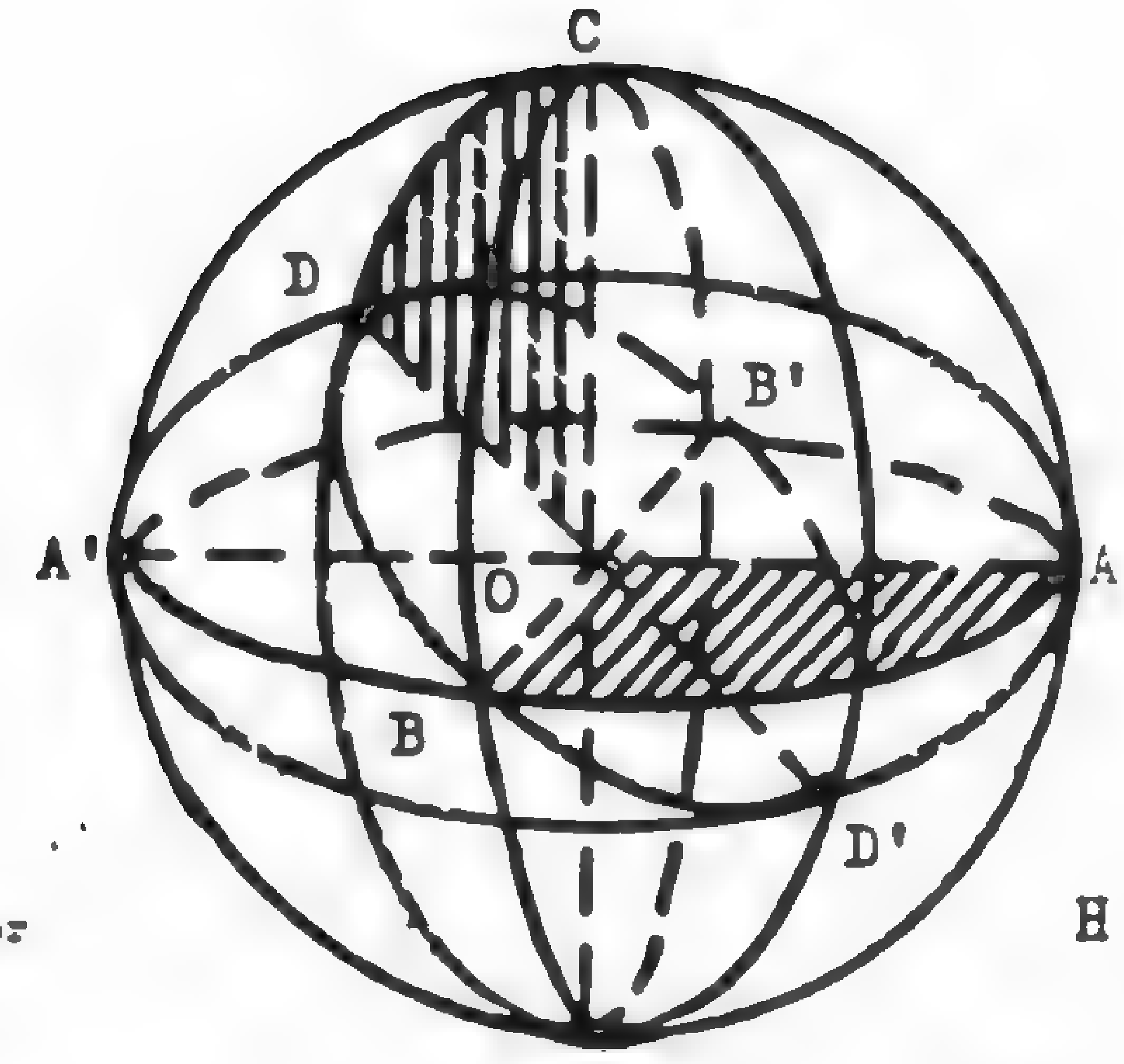


Fig. 70.

p-84, in the 1st-part of the sentence, "We shall call the boundaries where the 2 ½-hyperspheres H_B and $H_{B'}$ a FACE", read-as, "The great-sphere S_{ACD} of H which divides the rest of H into 2 ½-hyperspheres H_B and $H_{B'}$ is called the FACE of either of these ½-hyperspheres".

— CORRECTION — 6

65. VISIBLE AND HIDDEN-VIEWS IN THE HYPERSPHERE. The graphic-construction of the visible and hidden-views in a hypersphere are analogous to those of a sphere. Suppose for a moment, we considered only the hyperplane of the red-sphere S_{ACD}, then the red-sphere S_{ACD} will have all of its surface and interior-points visible, that is, any 3-space SOLID observed from a point outside of its hyperplane (not in the vanishing-plane of perspective) will have all its points visible. In the graphic-form of the hypersphere H, the hyperplane of the great-sphere S_{ACD} is a perpendicular-hyperplane to an observer lying outside of this hyperplane, but as we have the DOP-graphic, the axis DD' of the great-sphere S_{ACD} will be an oblique-axis, giving us the appearance of a great-sphere 'distorted' somewhat like our ordinary 3-space black-sphere in perspective. We shall have, then, a sphere S_{ACD} all in view, with no hidden-views whatsoever.

Now, if we take the ½-sphere $S^{A'}_{ACD}$ and rotate around the axis-plane of the great-circle (CDC'), the pole A' of $S^{A'}_{ACD}$ will rotate around the plane of the great-circle (CDC') along the path of its polar great-circle (ABA'), and we shall have the 2 arcs A'BA and AB'A' lying on opposite-sides of the hyperplane of the sphere S_{ACD}. The points B and B' are the poles of S_{ACD}; and the ½-hypersphere H_B will lie on one-side of the hyperplane of S_{ACD} and will be a visible-view in the DOP-graphic, that is, all of the ½-hypersurface of H_B will be visible, but its interior will be a hidden-view in the graphic. The other ½-hypersphere $H_{B'}$ will be a hidden-view together with its interior. The hypersphere H will then have the following visible and hidden-views in the DOP-graphic: all of the ½-hypersphere H_B will be a visible-view, but not its interior, only its ½-hypersurface will be visible; the ½-hypersphere $H_{B'}$ will be a hidden-view, with its ½-hypersurface and interior hidden-views.

The interior of the red-sphere S_{ACD} will be a hidden-view. We shall call the boundary where the 2 ½-hyperspheres H_B and $H_{B'}$ a FACE, which we may consider as belonging to either of these ½-hyperspheres (hyperhemispheres).

To see the above results in a clear way, compare the ½-spheres S^B_{AC} and $S^{B'}_{AC}$ of S_{ABC}, with their common-edge the great-circle (ACA'). The visible and hidden-views of H will then be analogous to the corresponding visible and hidden-views of the black-sphere S_{ABC}.

66. TRIRECTANGULAR SPHERICAL-TRIANGLES. THE VOLUME OF A ½-HYPERSPHERE. A spherical-triangle is called a TRIRECTANGULAR triangle if it has 3 right-angles, that is, its 3 sides are quadrants. Certain sets of trirectangular spherical-triangles on the hypersphere correspond somewhat to plano-conical-hypersurfaces, but restricted to portions of the hypersphere itself. In Fig. 70, if we take the trirectangular spherical-triangle A'CD, with its face as an element, and the great-circle (ABA') as directing-circle, and let CD be the spherical vertex-edge, then we can rotate around the fixed spherical vertex-edge CD and generate the volume of a ½-hypersphere. To see this, observe

that A'CD lies in the hyperplane of a sphere of revolution S_{ACD}. If we take the plane of the great-circle (CDC') as axis-plane and rotate A'CD around this axis-plane, with CD fixed, then A'CD and its face will generate the volume of a ½-hypersphere of H. Another way of seeing this, is to take the spherical-edge CD of A'CD as a spherical vertex-edge and rotate A'CD around CD as curved-axis, with the vertex-point A' opposite to CD lying on the directing-circle (ABA').

Since (CDC') and (ABA') are 2 polar great-circles, then CD will lie on (CDC'), with the vertex-point A' opposite to the spherical vertex-edge CD lying on (ABA'), and A' rotating around (CDC') on its polar great-circle (ABA'). The spherical-edge A'C will generate all of the surface of the ½-black-sphere S^C_{AB} except the pole C, and A'D will generate all of the surface of the ½-red-sphere S^D_{AB} except the pole D.

The student will find that the theorems of the double-cone can be modified to include a ½-hypersphere. In a way, we could call a ½-hypersphere a ½-DOUBLE-SPHERE, since the topological-structures between the double-cone and ½-double-sphere are the same, that is, the relationships of the parts to the whole between these 2 types of geometrical-structure. The 'genus' of these 2 types of geometrical-structure will be 0.

We have barely touched upon the development of the hypersphere, but considerable information can be gleaned at once from the graphic of the hypersphere and many new theorems are suggested by it. The student will find interesting applications of the trigonometry and calculus to the hyperspherical-geometry. We conclude this chapter by saying that, perphaps, the double-elliptic geometry of the hypersphere is the most interesting part of the higher-geometry of hyperspace.

[illegible]-Info! The [illegible] Hypersphere and Hypercone 1-21-77

George [illegible] Brandes

Dear Ray:

The H-books could go for $100.00/copy. Utterly fantastic... Many new-geometric-insights this week using the 4-space Synesthesia-geometry of hyperspace...I can't keep up with all the new geometrical-discoveries drawn from the H-book.

Presently working on developments of cut-a-way visible- and hidden-views in the hypersolid-figures in hyperspace. This week I discovered a small graphic-flaw in a visible-view within the visual-hypercube, the 2 canonical-hypercubes in the H-book (the only ones) should have the [illegible] axes as solid-lines, NOT dashed-lines. In any cell of the 4 visible-cubes of the hypercube, one can draw from any point within a visible-cell a ½-solid-line leaving the hypercube, and a ½-dashed-line entering the hypercube from the given point. In hyperspace, any point within a visible-cell can have a line pass through it, resulting in 2 ½-lines from the given point, 1 ½-line visible and 1 ½-line a hidden-view. However, if the line passes through the given point and lies in a hyperplane containing the given point, then the line becomes in its entirety a solid-line, assuming it lies in 1 of the visible-cells of the hypercube.

Many hyperspace-graphic laws can be derived from the visual-hypercube, which I am [illegible] to my astonishment. Indeed, the H-graphics were 100% and more.

Just a simple example: In 2-space Ax + By + K = 0 represents a line lying in the xy-plane, but in 3-space the above equation becomes the 'equation' of a plane perpendicular to the xy-plane and parallel to the z-axis. But consider the astounding 'equation' above in 4-space. The equation (same) Ax + By + K = 0 in 4-space has these [illegible]-properties: It represents the equation of a hyperplane perpendicular to [illegible] hyperplanes and passing through the given 'line' lying in their plane of [illegible], thus resulting in a hyperplane Ax + By + K = 0 in hyperspace such that [illegible] perpendicular to the xyz-hyperplane, xyw-hyperplane, and [illegible] coordinate-plane [illegible] except the zw-plane which is parallel to the hyperplane Ax + By + K = 0 in hyperspace. These results can be read-off directly from a mockup of a 4-space H-graphic with ease, then some.

4-space analytic-equations can be programmed directly into the H-graphics 100%, this is perhaps the most-astounding-property of the H-graphics, for the 4-space analytic-equations can be made visual for the 1st-time by assigning a Cartesian-coordinate-system to the canonical-hypercube, thus resulting in a [illegible] configuration of geometrical-points, etc.

Still more astounding-astounding, the visual double-cone and spherical-hypercone in hyperspace would put the Einstein-math in the [illegible]. The spherical-hypercone has 1 neg-dimension and 3 positive-dimensions, whereas in the double-cone, the one-cones each have respectively 1 neg-dimension and 2 pos-dimensions, ha! The spherical-hypercone H-graphic can be used to represent an infinity of time-tracks, where a single time-track can be represented on 1 x-cone...sections of a spherical-hypercone are [illegible] when a hyperplane passes through the vertex and center in the base of the spherical-hypercone, the base of a spherical-hypercone being the interior of a sphere. The different-sections of a spherical-hypercone represent different time-frames whenever 2 section-cuts result in different [illegible], i.e. x-cones with different radii in their [illegible]; when the x-cones have the same space-constant, then the time-tracks are equivalent in measurement. Other possibilities as to x-interpretations exist....

CHAPTER VII

EUCLIDEAN-GEOMETRY. FIGURES WITH PARALLEL ELEMENTS

67. THE AXIOM OF PARALLELS. We shall make a study of parallels and of figures with parallel elements in the Euclidean-geometry, and we shall assume the axiom of parallels.

Axiom. Through any point not a point of a given line passes 1 and only 1 line that lies in a plane with the given point and does not intersect it.

I. PARALLELS

68. PARALLEL LINES AND PARALLEL PLANES. THEOREMS ON PARALLELS. Lines and planes are parallel to one another as in the solid-geometry: 2 lines when they lie in 1 plane and do not intersect, a line and a plane or 2 planes when they lie in 1 hyperplane and do not intersect.

Theorem 1. 2 lines perpendicular to the same hyperplane are parallel.

Theorem 2. A hyperplane perpendicular to 1 of 2 parallel lines is perpendicular to the other.

Theorem 3. If 2 planes through a point are parallel to a given line they intersect in a parallel line.

Theorem 4. If a hyperplane intersects 1 of 2 parallel planes and does not contain it, the hyperplane intersects the other plane also, and the 2 lines of intersection are parallel.

For the hyperplane intersects the hyperplane of the parallel planes in a plane which intersects the parallel planes in parallel lines.

Theorem 5. If a plane meets 1 of 2 parallel planes in a single point, it will meet the other in a single point.

Theorem 6. 2 planes absolutely-perpendicular to a 3rd are parallel.

Theorem 7. A plane absolutely-perpendicular to 1 of 2 parallel planes is absolutely-perperpendicular to the other.

Theorem 8. 2 planes parallel to a 3rd are parallel to each other.

For a plane $\underline{Y}$ to the 3rd is $\underline{Y}$ to the 1st 2, and they are parallel by Th. 6.

Theorem 9. If 3 parallel planes all intersect a given line, they all lie in 1 hyperplane.

Theorem 10. 2 planes absolutely-perpendicular to 2 parallel planes are parallel, and 2 planes parallel respectively to 2 absolutely-perpendicular planes are absolutely-perpendicular.

Theorem 11. If 2 planes intersect in a line, planes through any point parallel to them intersect in a parallel line and form dihedral-angles equal to the dihedral-angles formed by the 2 given planes.

This theorem is proved by making use of Ths. 2 and 3. Corresponding plane-angles, and therefore corresponding dihedral-angles, are equal.

Corollary. If 2 planes are perpendicular, planes through any point parallel to them are also perpendicular.

Theorem 12. If 2 planes have a point in common, parallel planes through any other point make the same angles.

This theorem is proved by making use of Art. 38, Th. 11 and corollary. Corresponding plane-angles, and therefore corresponding dihedral-angles, are equal.

Corollary. A plane isocline to 1 of 2 parallel planes is isocline to the other and makes the same angles with both.

Theorem 13. 2 lines not in the same plane have only 1 common-perpendicular line.

Since the 2 lines lie in a hyperplane this is always a theorem of geometry of

3-dimensions, and is proved as a theorem of the solid-geometry.

Theorem 14. If a line and a plane do not lie in 1 hyperplane, they have only 1 common-perpendicular line. (See proof of Th. 1 of Art. 34.)

In some of the theorems given above, the following statements on dihedral-angles should be observed: CORRESPONDING DIHEDRAL-ANGLES have corresponding faces. 2 CORRESPONDING FACES are parallel $\frac{1}{2}$-planes lying in their hyperplane on the same-side of the plane determined by the 2 parallel lines.

The visual-graphics for the above theorems are so obvious that we had not considered them here. The student can make use of the graphic of the hypercube given in chapter I, and abstract-out the 'figures' associated with the theorems given above.

69. $\frac{1}{2}$-PARALLEL PLANES. 2 planes which do not lie in 1 hyperplane and do not intersect are said to be $\frac{1}{2}$-PARALLEL (semi-parallel).

Notation: The symbol for parallel is //; for $\frac{1}{2}$-parallel is $//_{\frac{1}{2}}$.

Theorem 1. The linear-elements of 2 $\frac{1}{2}$-parallel planes are all parallel to the other plane (see Art. 4[illegible]).

Theorem 2. The linear-elements which lie in 1 of 2 $\frac{1}{2}$-parallel planes are parallel to the other plane, and these are the only lines which lie in 1 plane and are parallel to the other.

Theorem 3. Through any point passes 1 and only 1 hyperplane perpendicular to each of 2 $\frac{1}{2}$-parallel planes (see Art. 26).

Theorem 4. 2 $\frac{1}{2}$-parallel planes have 1 and only 1 common-perpendicular plane (see Art. 34, Th. 2 and Fig. 50).

The proof for the 1st-part of this theorem follows from Th. 2 of Art. 34, that is, there is 1 common-perpendicular plane. For the 2nd-part of the theorem we prove that there is only 1 common-perpendicular plane, we take a plane perpendicular to the 2 given half-parallel planes and prove that this plane coincides with the plane determined as in Th. 2 of Art. 34. In Fig. 50, the planes α and β are $\frac{1}{2}$-parallel, that is, $\alpha //_{\frac{1}{2}} \beta$; the common-perpendicular plane is γ .

Theorem 5. The only common-perpendicular lines of 2 $\frac{1}{2}$-parallel planes are those which lie in the common-perpendicular plane.

The PERPENDICULAR-DISTANCE or simply the DISTANCE, between 2 $\frac{1}{2}$-parallel planes is the distance between the points where they are cut by a common-perpendicular line. It is the same for all of these lines, since the common-perpendicular plane cuts the given plane in parallel lines (Th. 1).

Theorem 6. The perpendicular-distance between 2 $\frac{1}{2}$-parallel planes is less than the distance measured along any line which intersects both and is not perpendicular to both.

Theorem 7. 2 planes through a point parallel respectively to 2 $\frac{1}{2}$-parallel planes intersect in a line which is parallel to their linear-elements.

For the line through the point parallel to the linear-elements is parallel to the 2 given planes, and therefore lies in both of the 2 planes which are parallel to them through the point.

Theorem 8. If a plane distinct from each of 2 parallel planes intersects 1 in a line and does not intersect the other in a line, it will be $\frac{1}{2}$-parallel to the 2nd. (Fig. 71.)

Given: A plane γ distinct from each of 2 // planes α and β , with γ intersecting α in a line c and which does not intersect β .

To Prove: $\gamma //_{\frac{1}{2}} \beta$.

Proof: If the plane γ were in a hyperplane with β , this hyperplane, containing the line c in which γ intersects α , must be the hyperplane of the // planes α and β ; or if γ intersected the // plane β in a point, it would lie entirely in the hyperplane of the // planes α and β . Thus, in either case, we would have a plane γ lying in the hyperplane

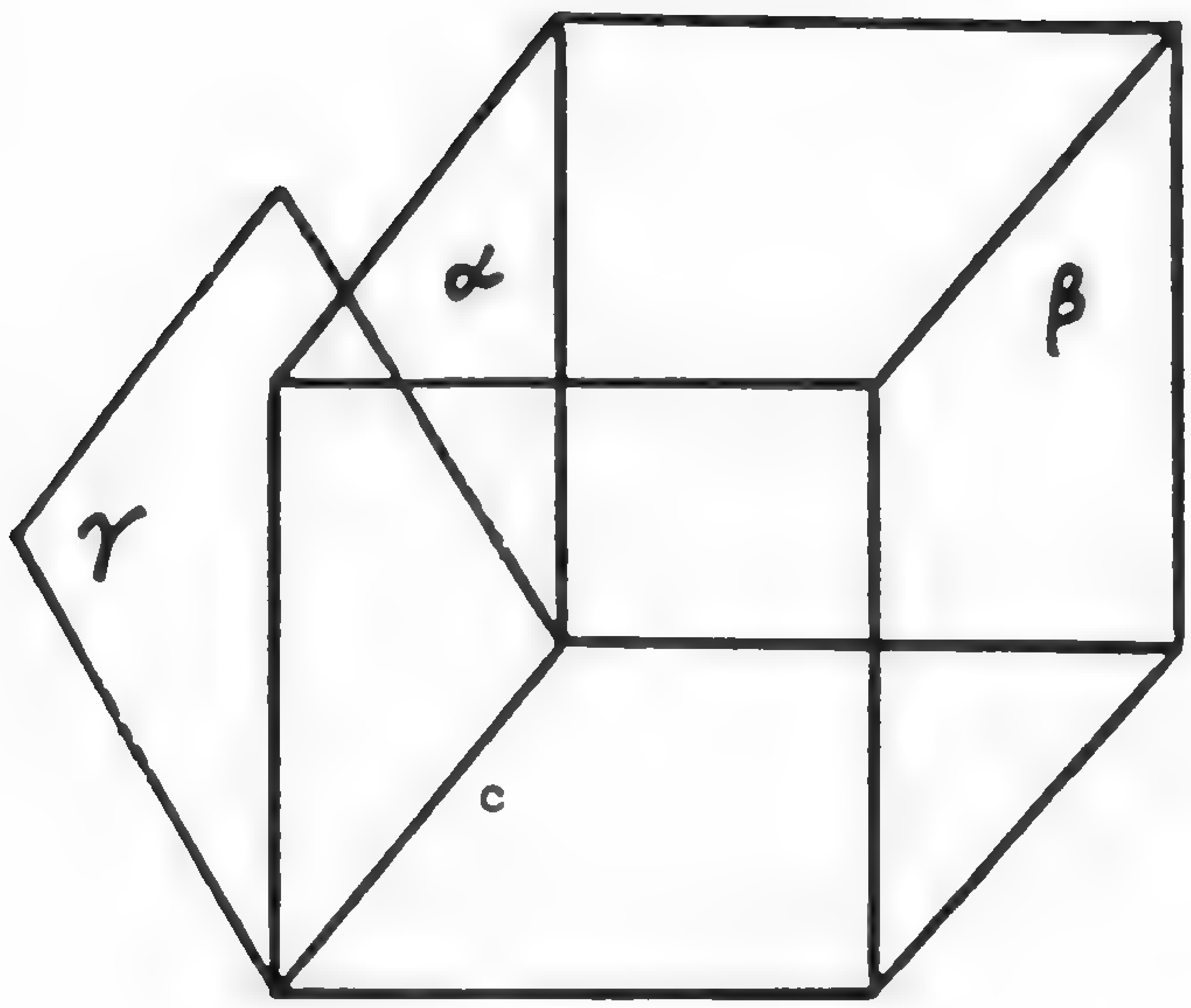


Fig. 71.

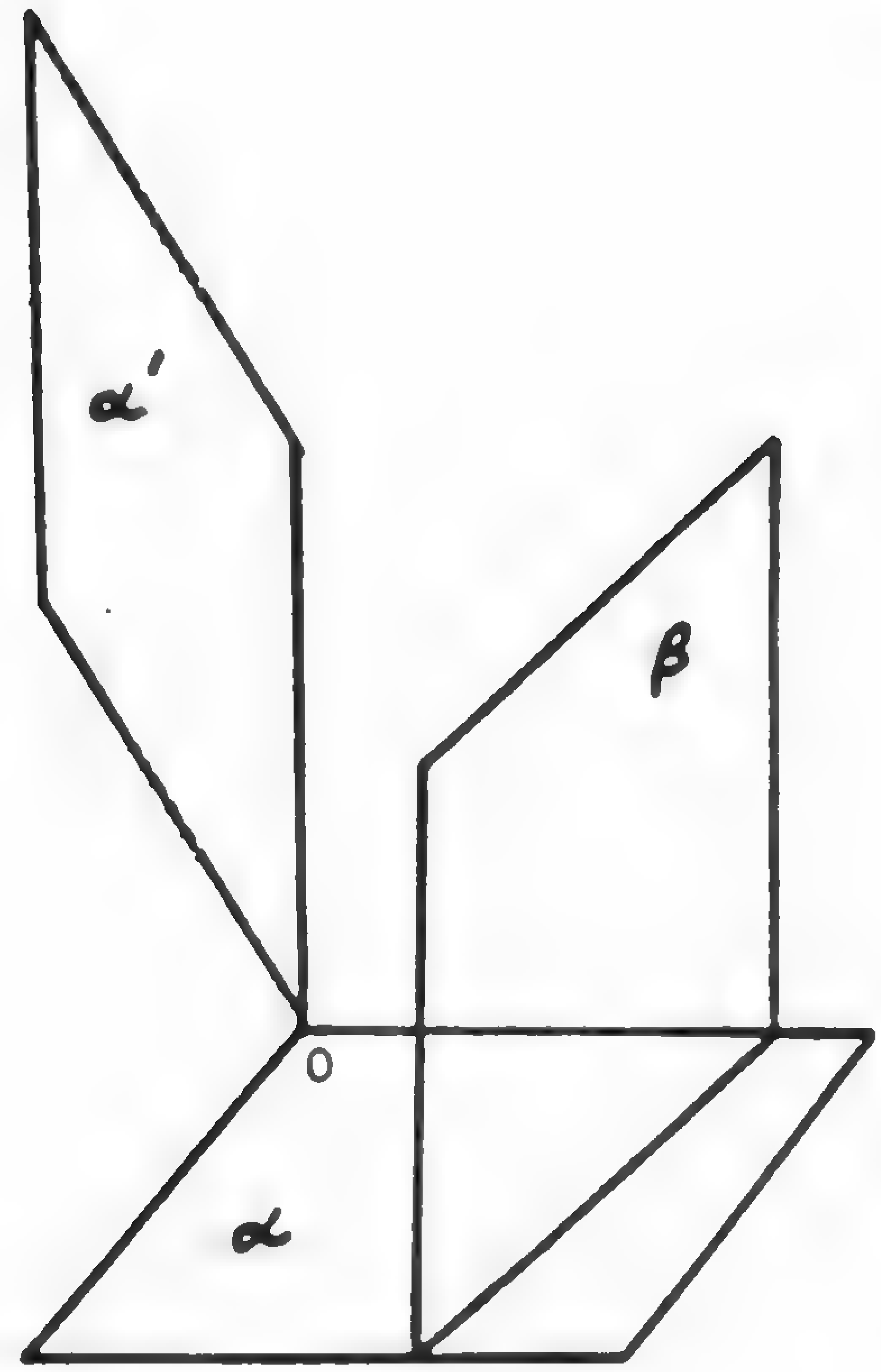


Fig. 72.

of the 2 // planes α and β, intersecting α in a line, and therefore β in a line. As the plane γ does not intersect the // plane β in a line, it cannot lie in a hyperplane with it nor intersect it at all. Therefore $\gamma //_{\frac{1}{2}} \beta$. (Q.E.D)

Theorem 9. If a plane perpendicular to 1 of 2 absolutely-perpendicular planes does not contain their point of intersection, it is $\frac{1}{2}$-parallel to the other. (Fig. 72.)

Given: 2 $\underline{\vee}$ planes α and α' which intersect at a point O, and β a plane $\perp$ to α but not containing O.

To Prove: $\beta //_{\frac{1}{2}} \alpha'$.

Proof: β cannot lie in a hyperplane with α', for such a hyperplane would intersect α only in a line through O. Nor can β intersect α' even in a point, for then it would contain the line through such a point $\perp$ to α, and also contain the point O. Therefore $\beta //_{\frac{1}{2}} \alpha'$. (Q.E.D)

70. THEOREMS ON LINES AND PLANES PARALLEL TO A HYPERPLANE AND PARALLEL HYPERPLANES. A line and a hyperplane, a plane and a hyperplane, or 2 hyperplanes are PARALLEL when they do not intersect.

Theorem 1. If a line, not a line of a given hyperplane, is parallel to a line of the hyperplane, it is parallel to the hyperplane; and if a plane, not a plane of a given hyperplane, is parallel to a plane of the hyperplane, it is parallel to the hyperplane.

Theorem 2. If a line is parallel to a hyperplane, it is parallel to the intersection of the hyperplane with any plane through it or with any hyperplane through it; and if a plane is parallel to a hyperplane, it is parallel to the intersection of the hyperplane with any hyperplane through it.

Theorem 3. If a line is parallel to a hyperplane, a line through any point of the hyperplane parallel to the given line lies wholly in the hyperplane; and if a plane is parallel to a hyperplane, a plane or line through any point of the hyperplane parallel to the given plane lies wholly in the hyperplane.

Theorem. 4. 2 hyperplanes perpendicular to the same line are parallel.

Theorem 5. If 1 of 2 parallel hyperplanes is perpendicular to a line, the other is also perpendicular to the line.

Theorem 6. Through a point not a point of a given hyperplane, can be passed 1 and only 1 parallel hyperplane.

Theorem 7. All the lines and planes in 1 of 2 parallel hyperplanes are parallel to the other, and all the lines and planes through a point, parallel to a hyperplane, lie in a parallel hyperplane.

Theorem 8. If a plane intersects 2 parallel hyperplanes, or if a hyperplane intersects 2 parallel planes, the lines of intersection are parallel; and if a hyperplane intersects 2 parallel hyperplanes, the planes of intersection are parallel.

Theorem 9. If 3 non-coplanar lines through a point are respectively parallel to 3 other non-coplanar lines through a point, the 2 sets of lines determine the same hyperplane or parallel hyperplanes; or if an intersecting line and plane are respectively parallel to another intersecting line and plane, they determine the same hyperplane or parallel hyperplanes.

Theorem 10. 2 trihedral-angles having their sides parallel each to each and extending in the same direction from their vertices are congruent.

For the corresponding face-angles are equal.
In regards to the theorem, it should be observed that 2 parallel $\frac{1}{2}$-lines extend in the same-direction when in their plane they lie on the same-side of the line determined by their extremities.

Theorem 11. If a line is parallel to a hyperplane, all points of the line are at the same-distance from the hyperplane; or if a plane is parallel to a hyperplane, all points of the plane are at the same-distance from the hyperplane.

Theorem 12. 2 parallel hyperplanes are everywhere equidistant.

For the graphics associated with the above theorems, use the graphic of the hypercube given in chapter I, and abstract-out the corresponding relationships in pictorial-form.

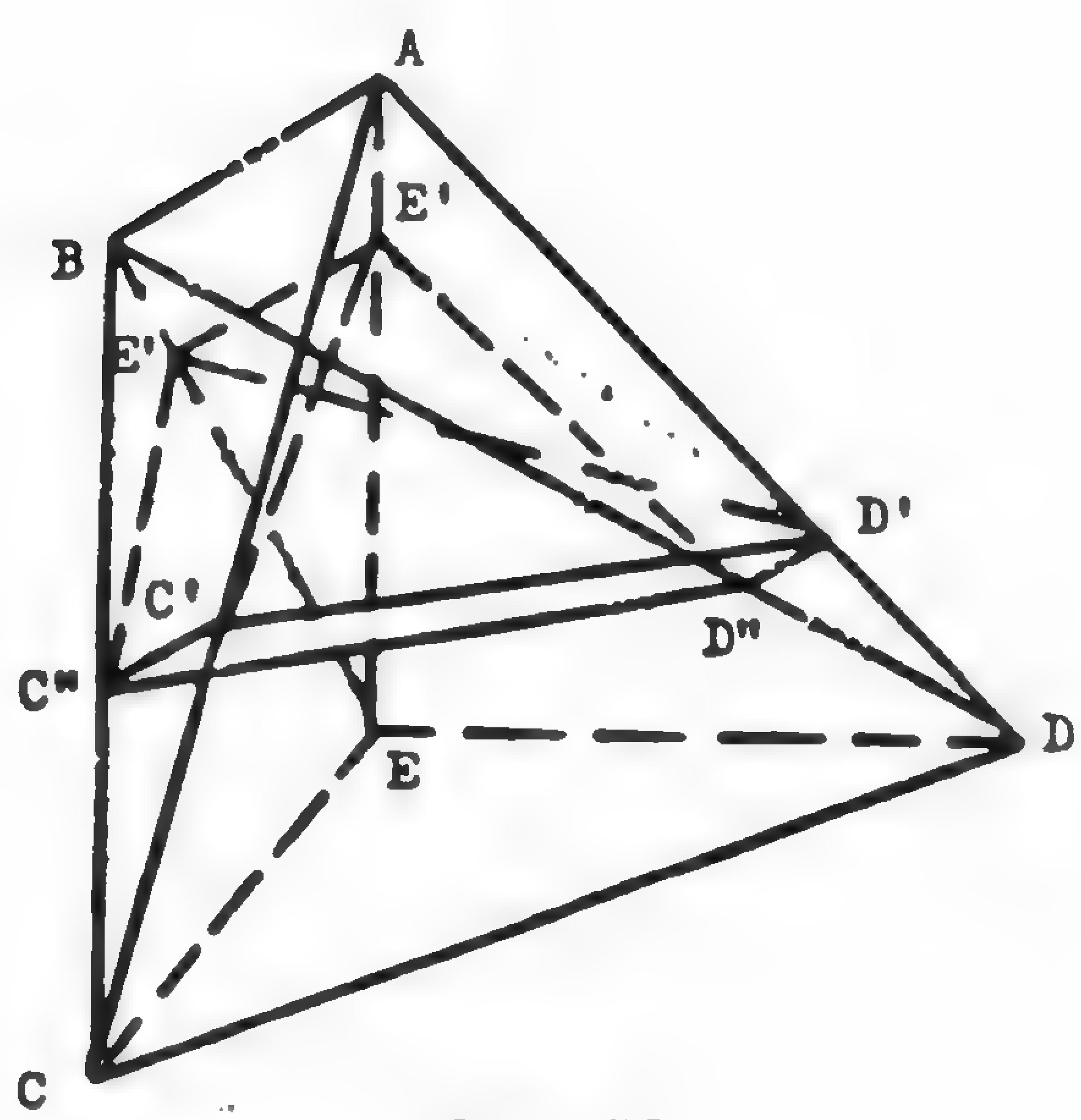


Fig. 73.

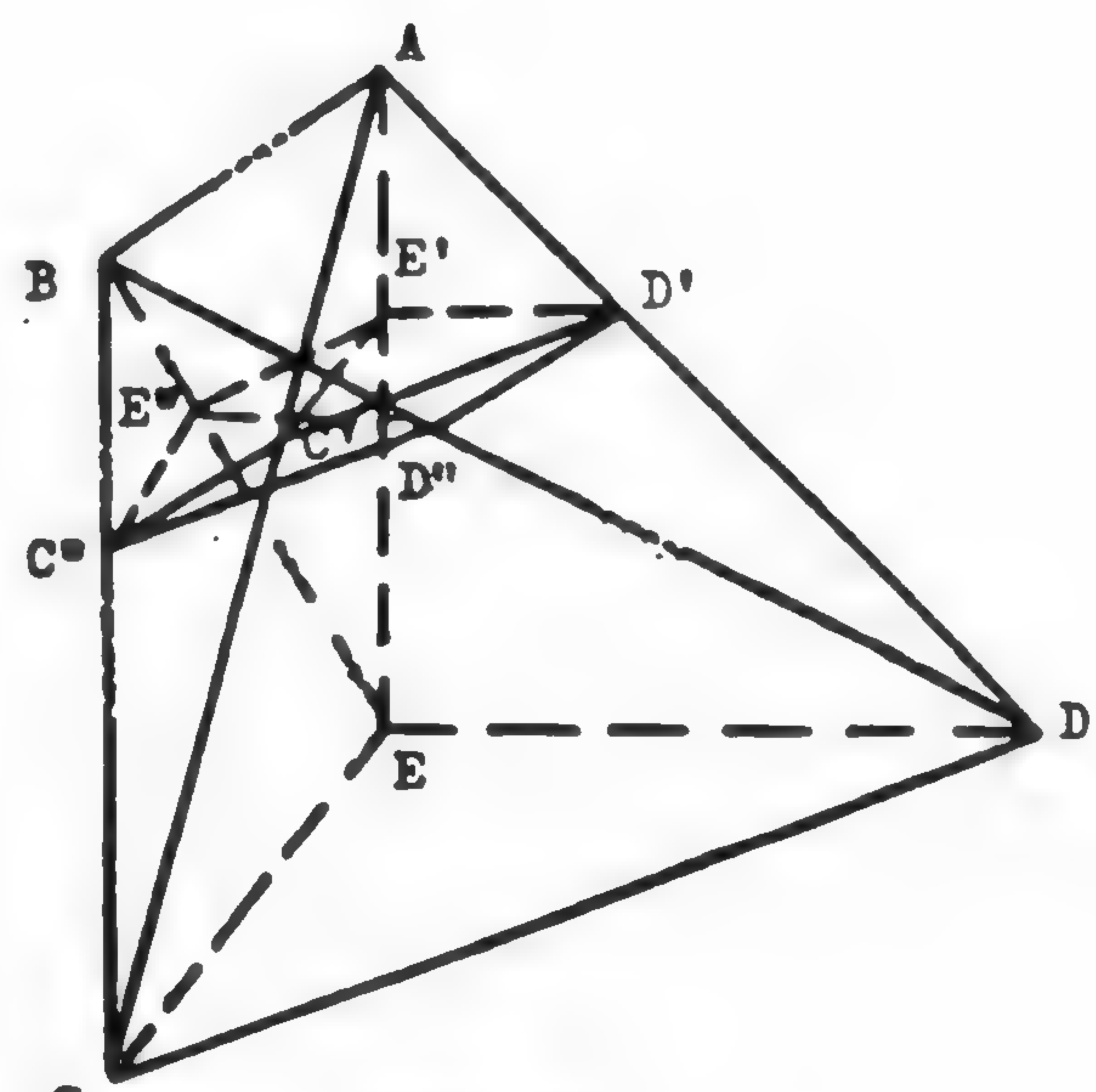


Fig. 74.

p-89, Fig. 73, the dashed red-line E'D' should be removed and replaced by the dashed red-line E'D"; that is, a dashed red-line drawn through the 2 points E' and D", and not through E' and D' as shown in the hyperspace-figure. CORRECTION PAGE —

Theorem. Let ABCDE be a pentahedroid cut by a hyperplane α so that the edge AB lies on one side of α and the face CDE on the other side. Then if α is parallel to the line AB and to the plane CDE, the section will be a prism; if α is parallel to the line but not to the plane, the section will be a truncated-prism; if α is parallel to the plane but not to the line, the section will be frustrum of a pyramid; or if α is not parallel to the line nor to the plane, the section will be a truncated-pyramid (see Art. 76, Th. 3).

The theorem is proved by considering the plane-sections of its cells. The graphics for the 4 cases of the theorem are as follows:

Case 1. If α is parallel to the line AB and to the plane CDE, the section will be a prism C'D'E'-C"D"E". (Fig. 74.)

Case 2. If α is parallel to the line AB but not to the plane CDE, the section will be a truncated-prism C'D'E'-C"D"E". (Fig. 73.)

Case 3. If α is parallel to the plane CDE but not to the line AB, the section will be a frustrum of a pyramid C'D'E'-C"D"E". (see Fig. 5.)

Case 4. If α is not parallel to the line AB nor to the plane CDE, the section will be a truncated-pyramid. The section will be somewhat like Fig. 5, with C'D'E' and C"D"E" not parallel to CDE.

71. ISOCLINE-PROJECTION. AREA OF AN ISOCLINE-PROJECTION OF A PLANE-POLYGON.

Theorem 1. Any plane-polygon and its projection upon an isocline-plane are similar. (Fig. 75, Logic-Diag. 10.)

Given: Any plane-polygon P_n of n-sides, and P'_n its projection upon an isocline-plane.

To Prove: $P_n \sim P'_n$.

Proof: Let O be the point of intersection of 2 isocline-planes bc and pq. Let the plane-polygon P_n lie in pq, and let bc be the isocline-plane upon which the points of P_n are projected. In pq, take any 2 points E and F, and E' and F' their projections upon bc.

To determine the points E' and F', use Logic-Diag. 10, and the following graphic-construction: let the angles between the 2 isocline-planes bc and pq be ϕ; in pq, the $\frac{1}{2}$-lines e and f make with the $\frac{1}{2}$-line p, the angles δ and γ respectively, now lay-off these 2 acute-angles in bc and ad, and form the 2 common $\perp$ planes e'e" and f'f" which project the points e and f upon bc; since the common $\perp$ plane f'f" intersects pq in the $\frac{1}{2}$-line f, the point F will lie in f'f", now draw a line in f'f" parallel to the $\frac{1}{2}$-line f" and passing through the point F, this line will intersect the $\frac{1}{2}$-line f' in a point F', and in f'f", the line-segment FF' will be $\perp$ to the $\frac{1}{2}$-line f' at F' forming a right-angle OF'F; the angle FOF' = ϕ, since a common $\perp$ plane of bc and pq will cut-out the same angle ϕ as ba and cd does on pq and bc respectively; now form a 2nd common $\perp$ plane e'e" to bc and pq and repeat the process used above for finding the projection of a point upon an isocline-plane, then angle EOE' = ϕ, and angle OE'E is a right-angle; the angle EOF = angle E'OF' by construction, and we shall denote this angle by θ (Theta).

OE'E and OF'F are 2 right-triangles with equal acute-angles θ at O. They are similar (2 right-triangles are similar if an acute-angle of one equals an acute-angle of the other.), and the sides OE and OF are proportional to the sides OE' and OF'. The angle EOF = angle E'OF' = θ. Therefore the triangles OEF and OE'F' are themselves similar (2 triangles are similar if an angle of one equals an angle of the other and the sides including these angles are in proportion.). Now if the triangles formed by joining the vertices of a plane-polygon P_n to a point O in its plane are respectively similar to the triangles formed in the same way from another polygon P'_n, the 2 polygons are similar. Therefore $P_n \sim P'_n$. (Q.E.D)

In Fig. 75, let A be the area of the triangle OEF, and let A' be the area of the triangle OE'F', then $A = \frac{OE.OF}{2} \sin EOF$, and $A' = \frac{OE'.OF'}{2} \sin E'OF'$, since the area of a triangle equals one-half of the product of any 2 adjacent-sides and the included angle. Now angle EOF = angle E'OF' = θ, and we have

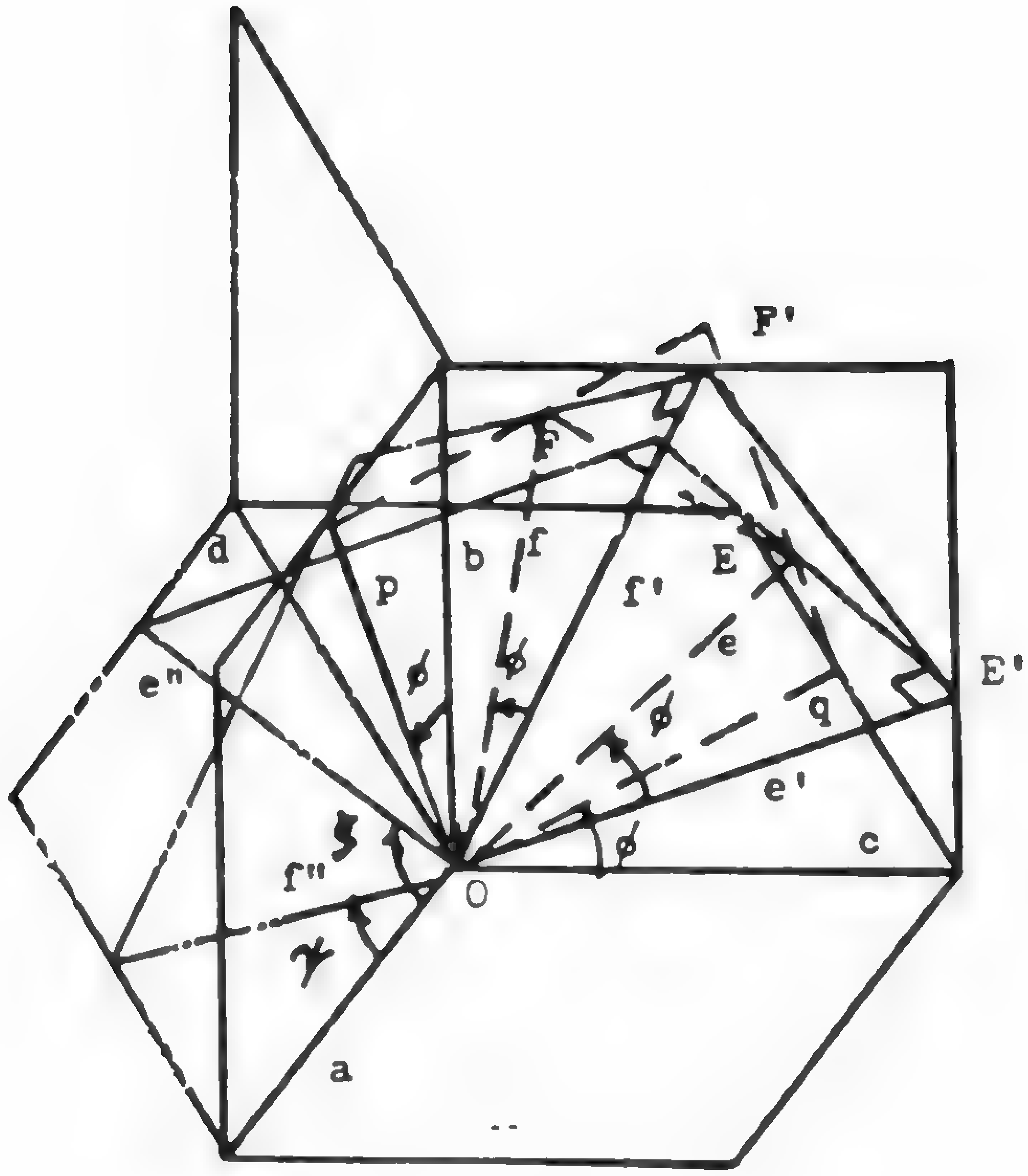
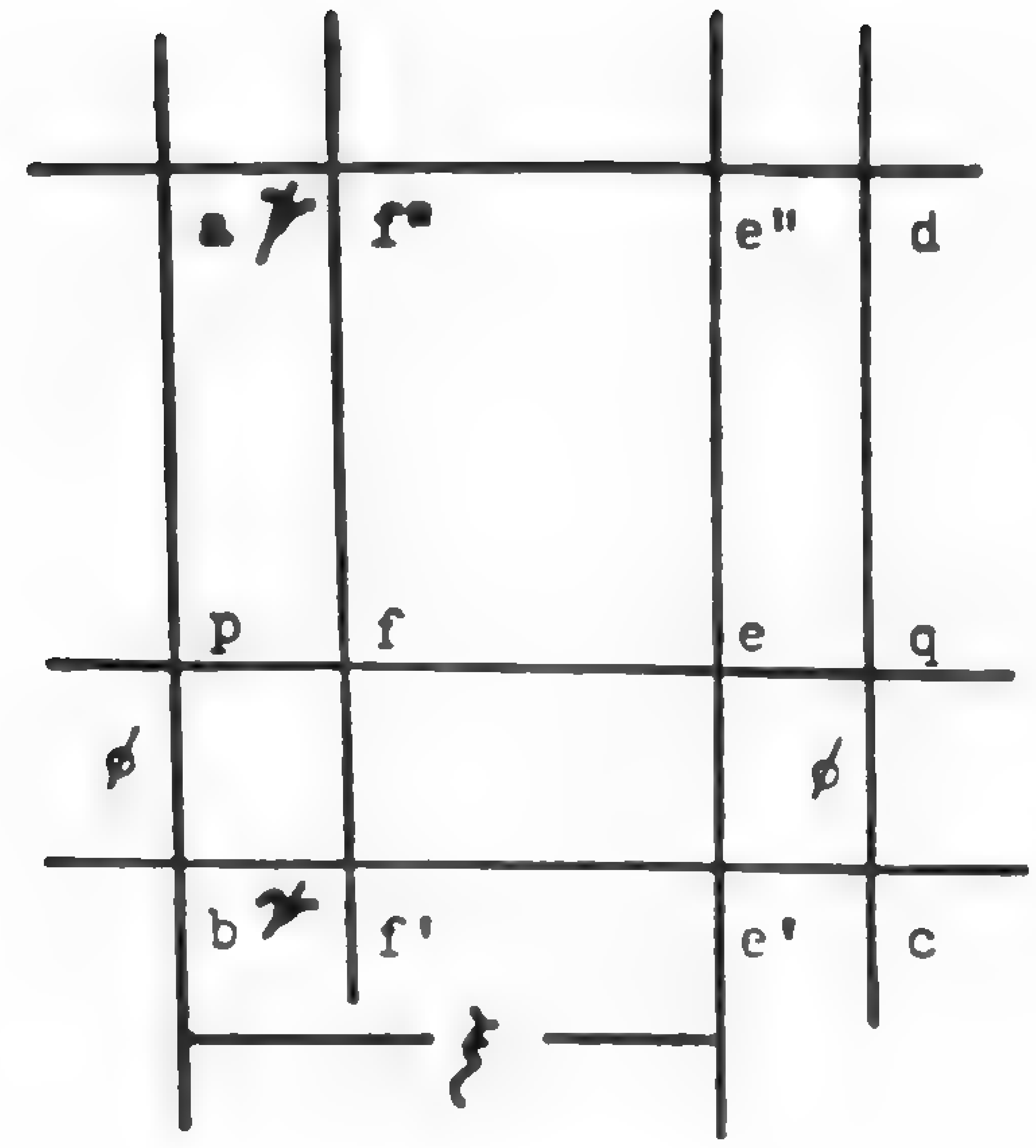


Fig. 74.



Logic-Diag. 10.

$A = \frac{OE.OF}{2} \sin \theta$, and $A' = \frac{OE'.OF'}{2} \sin \theta'$ $(\theta' = \theta)$; and we have

$A' = \frac{OE'.OF'}{2} \sin \theta$. Since OF'F and OE'E are right-triangles, we have

$OE' = OE \cos \phi$, and $OF' = OF \cos \phi$, substituting these expressions on the right-hand side of A', gives us

$A' = \frac{OE.OF}{2} \sin \theta \cos \phi \cos \phi$. Substituting A for $\frac{OE.OF}{2} \sin \theta$ on the right-hand side of A', gives us $A' = A \cos \phi \cos \phi$, which simplifies to $A' = A \cos^2 \phi$.

Corollary. The projection of a circle upon a plane isocline to its plane is a circle.

Theorem 2. Conversely, if a plane-polygon is similar to its projection upon another plane, the 2 planes are isocline or parallel.

72. PRISMOIDAL-HYPERSURFACES. INTERIORS. SECTIONS. AXES. A PRISMOIDAL-HYPERSURFACE consists of a system of parallel lines passing through the points of a given polyhedron but not lying in the hyperplane of the polyhedron. The polyhedron is called the DIRECTING-POLYHEDRON, the parallel lines are the ELEMENTS, and the elements which pass through the vertices are LATERAL-EDGES. We shall consider only directing-polyhedrons which are simple and convex.

The elements which pass through the points of a face of the directing-polyhedron constitute the interior of a prismoidal-surface and a CELL of the hypersurface. The elements which pass through the points of an edge of the directing-polyhedron constitute a face of a prismoidal-surface and a STRIP of the hypersurface. A strip is that portion of a plane lying between 2 parallel lines, and is a face of the hypersurface, the common -face of 2 adjacent prismatic-surfaces.

The INTERIOR OF A PRISMOIDAL-HYPERSURFACE consists of the lines which pass through the points of the interior of the directing-polyhedron and are parallel to the elements. The interior of any segment whose points are points of the hypersurface (being convex) will lie entirely in the interior of the hypersurface unless it lies entirely in the hypersurface itself, and a $\frac{1}{2}$-line drawn from a point of the interior and not parallel to the elements will intersect the hypersurface in 1 and only 1 point.

Fig. 76 represents the graphic-figure of a tetrahedroidal prismoidal-hypersurface. It is the simplest prismoidal-hypersurface. In Fig. 76, we have not taken any points in the interior of the hypersurface nor the interior points of the directing-tetrahedron—in the next section when we make a study of the hyperprism, the graphics will include the interior points of a restricted-portion of a prismoidal-hypersurface as well as the interior points of its directing-polyhedron. The elements in the graphic can be considered either as element-segments generating a portion of the hypersurface or as elements of the hypersurface itself, that is, element-segments can be taken as 'lines' in the graphic—compare this to the graphic-representation of prismoidal-surfaces in the solid-geometry.

In Fig. 76, the tetrahedron ABCD is a directing-tetrahedron of the tetrahedroidal prismoidal-hypersurface. Its lateral-edges are AA', BB', CC', and DD'. Its 4 prismatic-surfaces are A'B'D'-ABD, A'C'D'-ACD, B'C'D'-BCD, and A'B'C'-ABC. The interiors of these prismatic-surfaces are its cells. Its 6 faces are the strips A'D'-AD, B'D'-BD, C'D'-CD, A'B'-AB, B'G'-BC, and C'A'-CA.

The tetrahedroidal prisomoidal-hypersurface is the analogue to the triangular-prismatic -surface in the solid-geometry: compare the triangular prismatic-surface B'C'D'-BCD in Fig. 76 to the tetrahedroidal prismoidal-hypersurface itself.

The visible and hidden-views of the hypersurface correspond somewhat like that of the prismatic-surface B'C'D'-BCD in its hyperplane. The cell A'B'C'-ABC is a visible-view in the graphic as well as portions of the 3 other cells of the hypersurface. Since the directing-tetrahedron ABCD is a hyperplane-tetrahedron enclosing a portion of its hyperplane, we can consider the tetrahedroidal prismoidal-hypersurface a closed-hypersurface.

Theorem 1. A hyperplane passing through a point of the interior of a prismoidal-hypersurface and parallel to the elements intersects the hypersurface in a prismatic-surface. (Fig. 77.)

For a hyperplane intersects the directing-polyhedron in a convex-polygon, and intersects the hypersurface in the elements which pass through the points of this polygon.

Theorem 2. A hyperplane which is not parallel to the elements of a prismoidal-hypersurface intersects the hypersurface in a polyhedron, and any such polyhedron can be taken as directing-polyhedron.

A RIGHT-SECTION is a directing-polyhedron whose hyperplane is perpendicular to the elements.

Theorem 3. Directing-polyhedrons lying in parallel hyperplanes are congruent, and any 2 homologous points of 2 such polyhedrons lie in a line parallel to the elements. (see Fig. 76.)

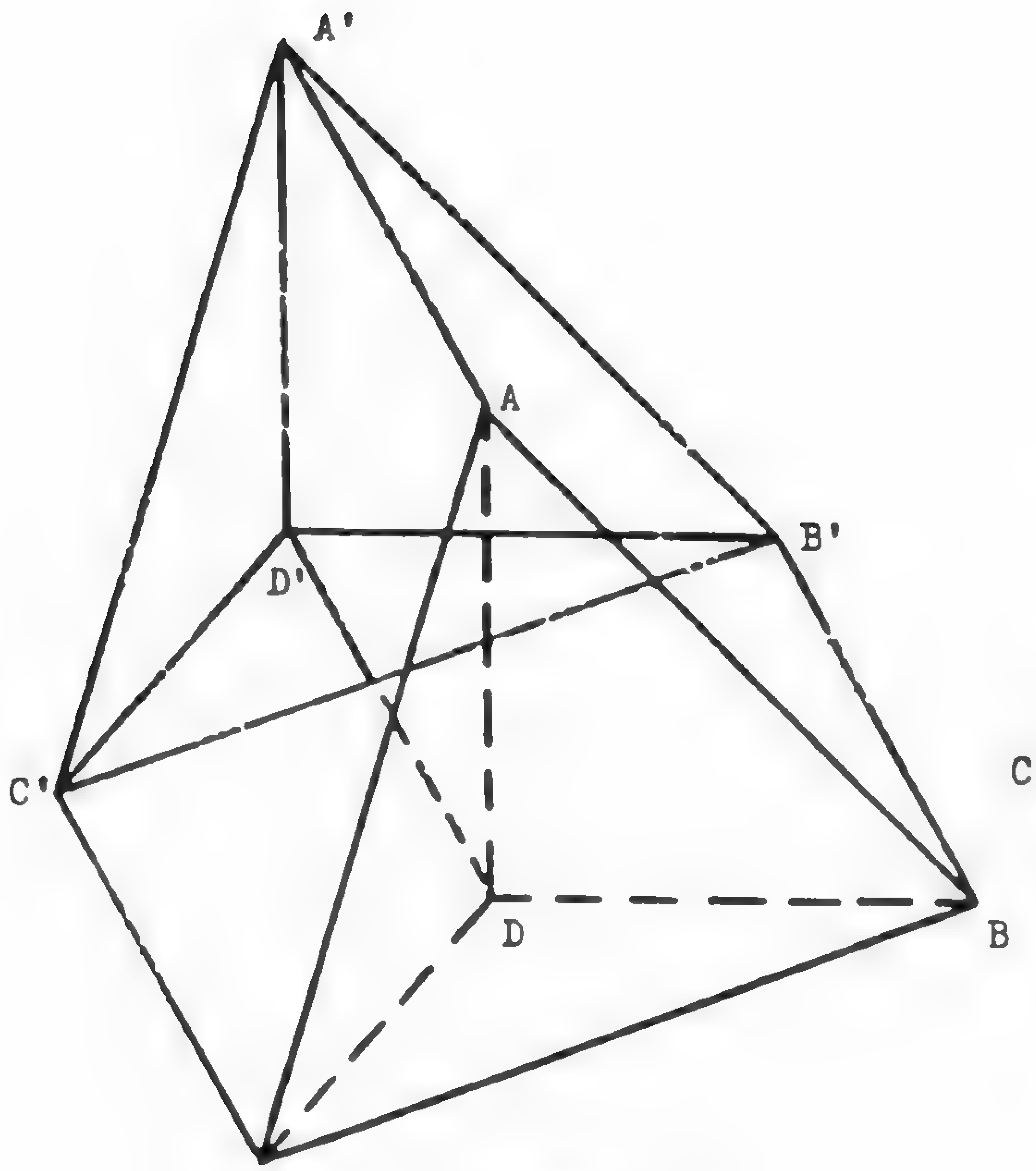


Fig. 76.

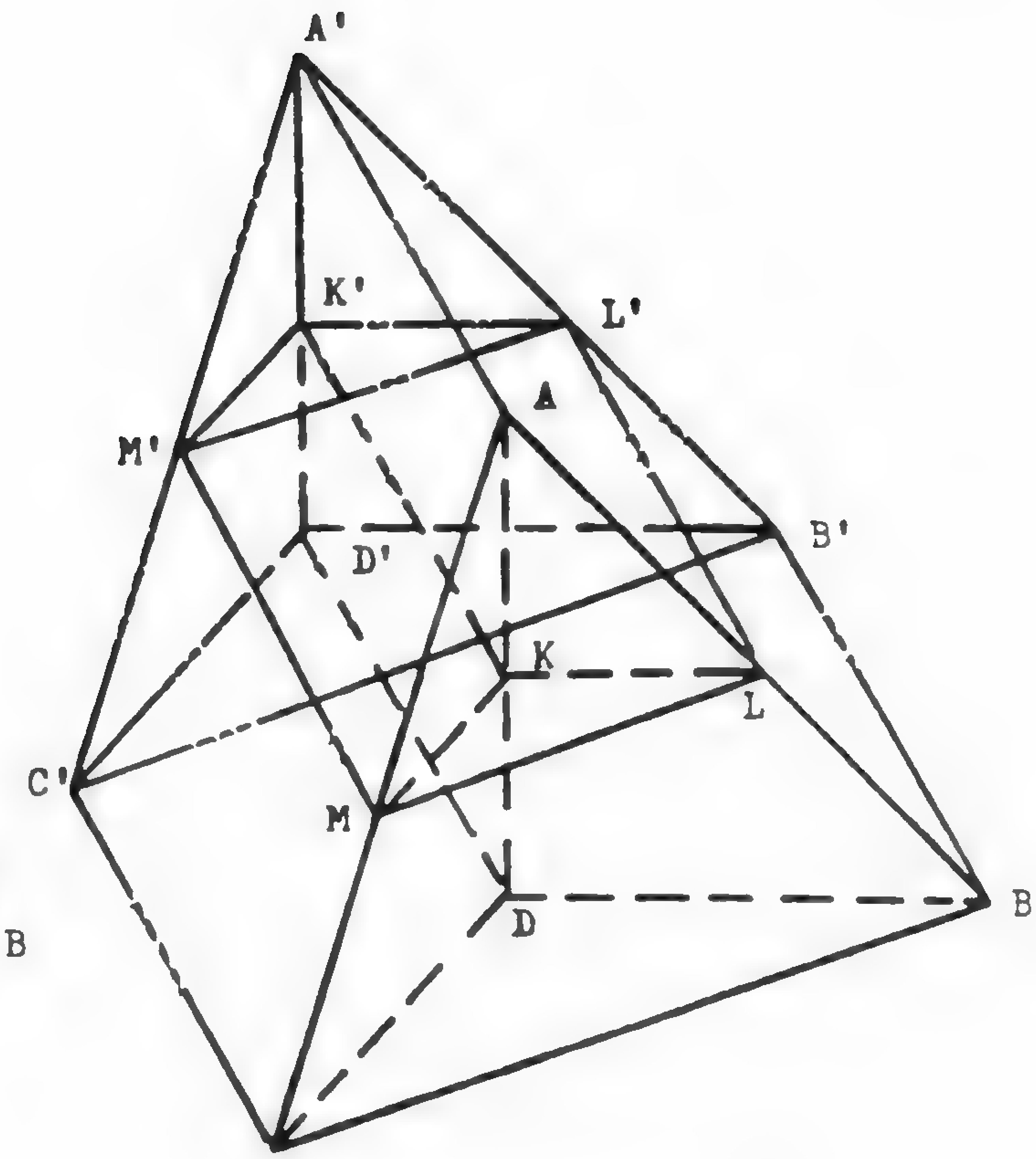


Fig. 77.

Theorem 4. If a prismoidal-hypersurface has a parallelopiped for directing-polyhedron, it will have 3 pairs of equal opposite lateral-cells lying in parallel hyperplanes, and all of its directing-polyhedrons will be parallelopipeds. (The hypercube is a special-case of the theorem.)

The proof of the theorem follows from the fact that any 2 opposite-faces of the given parallelopiped are equal parallelograms lying in parallel planes, and are directing-polygons of equal prismatic-surfaces lying in parallel hyperplanes (Art. 70, Th. 9). Therefore any directing-polyhedron will have 3 pairs of parallel opposite-faces (Art. 70, Th. 8), and will be a parallelopiped.

Theorem 5. If any directing-polyhedron of a prismoidal-hypersurface has a center of symmetry, the line through this point parallel to the elements is an axis of symmetry, meeting the hyperplane of every directing-polyhedron in a point which is a center of symmetry of this polyhedron. Each point of the line is, in fact, a center of symmetry for the entire hypersurface, and the line as-a-whole is a line of symmetry. (Fig. 78.)

For this line lies mid-way between the 2 lines in which any plane containing it intersects the hypersurface, and any line intersecting it determines with it such a plane.

In Fig. 78, let the point P be the center of symmetry of a directing-tetrahedron ABCD, and let PP' be the line through P parallel to the elements. Then PP' is an axis of symmetry, meeting the hyperplane of every directing-tetrahedron in a point which is a center of symmetry of this tetrahedron. For example, the axis PP' meets the hyperplane of the directing-tetrahedron A'B'C'D' at its center of symmetry in the point P'.

73. HYPERPRISMS. INTERIOR OF A HYPERPRISM. A HYPERPRISM consists of that portion of a prismoidal-hypersurface which lies between 2 parallel directing-polyhedrons, together with the directing-polyhedrons themselves and their interiors.

The interiors of the directing-polyhedrons are the BASES. In each hyperplane of the hypersurface we have a prism whose interior is 1 of the LATERAL-CELLS of the hyperprism. The lateral-faces and edges of these prisms are the LATERAL-FACES and LATERAL-EDGES of the hyperprism. The lateral-edges are all equal; the bases are congruent (Art. 72, Th. 3).

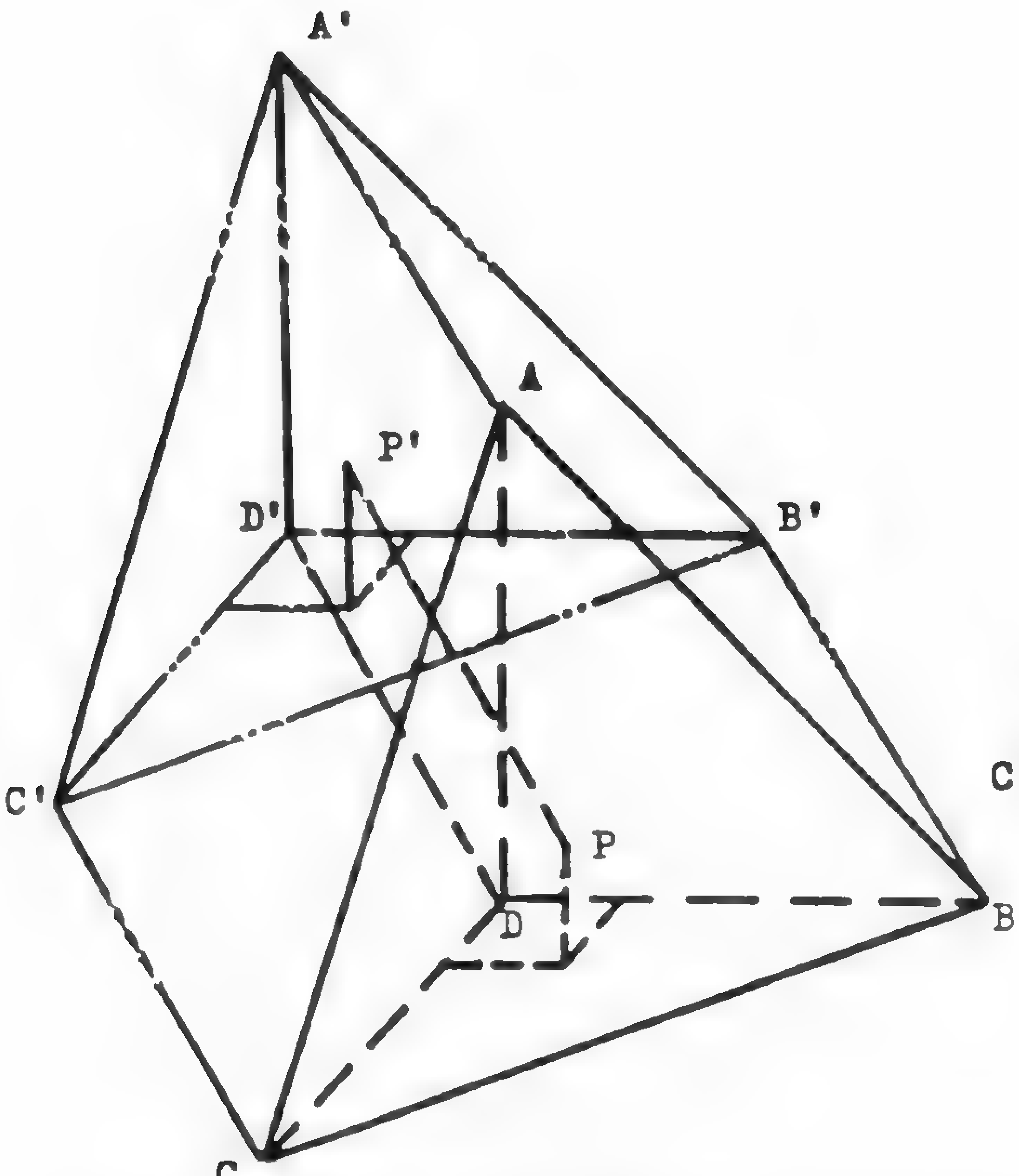


Fig. 78.

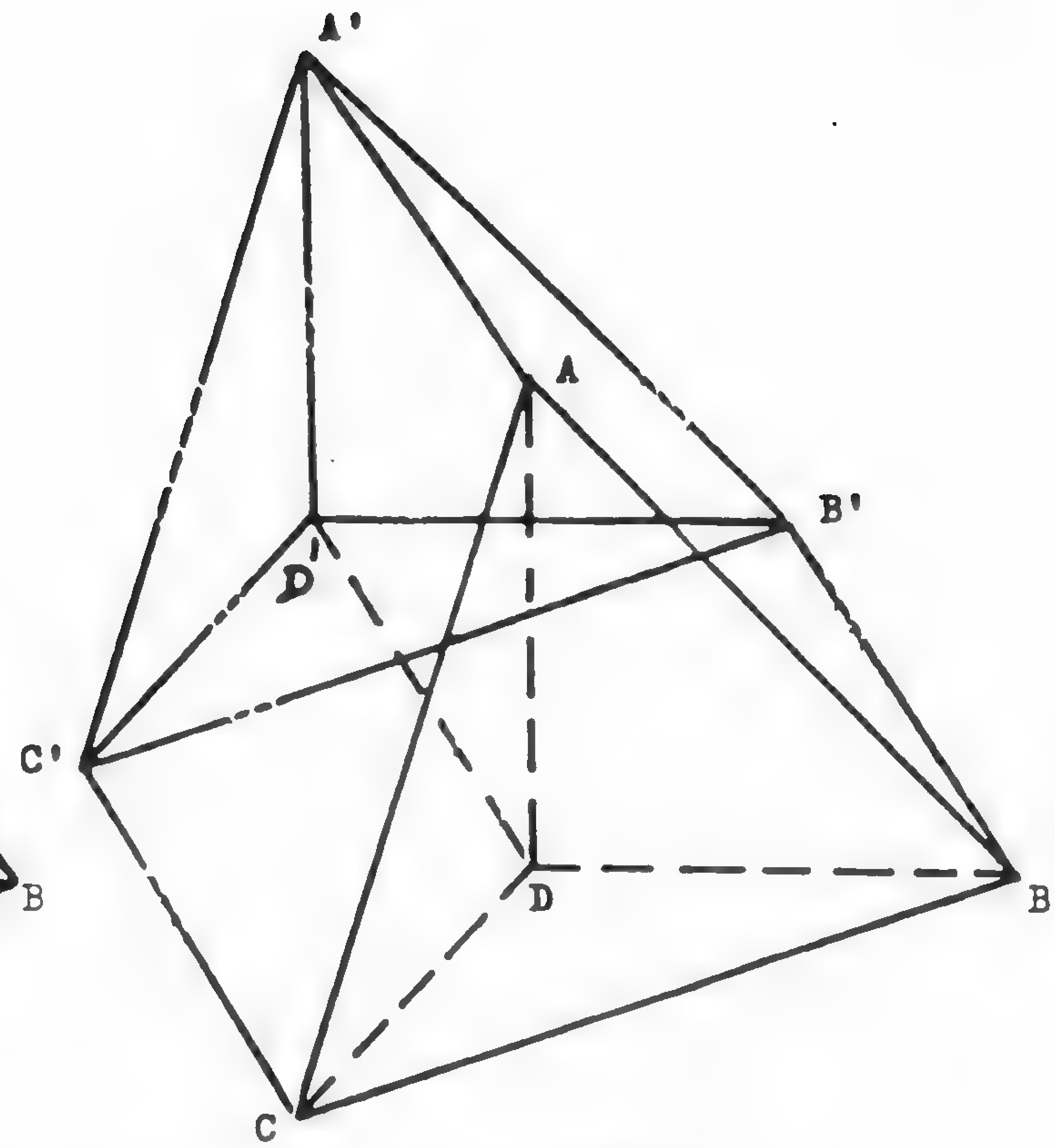


Fig. 79.

The INTERIOR OF A HYPERPRISM consists of that portion of the interior of the prismodal-hypersurface which lies between the bases. The hypersurface being convex, the interior of any segment whose points are points of the hyperprism will lie entirely in the interior of the hyperprism unless it lies entirely in the hyperprism itself, and a $\frac{1}{2}$-line drawn from a point of the interior will intersect the hyperprism in 1 and only 1 point.

A hyperprism is a RIGHT HYPERPRISM when the lateral-edges are perpendicular to the hyperplanes of the bases. When the bases are the interiors of regular-polyhedrons the hyperprism is REGULAR.

Fig. 79 represents a tetrahedroidal-hyperprism which may be denoted by A'B'C'D'-ABCD. The interiors of the 2 directing-tetrahedrons A'B'C'D' and ABCD are its bases. It has 4 lateral-cells which are the interiors of the 4 prisms A'B'D'-ABD, A'C'D'-ACD, B'C'D'-BCD, and A'B'C'-ABC. The lateral-faces and edges of these 4 prisms are the lateral-faces and lateral-edges of A'B'C'D'-ABCD. Its 4 lateral-edges A'A, B'B, C'C, and D'D are all equal; the bases A'B'C'D' and ABCD are congruent.

The interior of the tetrahedroidal-hyperprism A'B'C'D'-ABCD consists of that portion of the interior of the tetrahedroidal prismoidal-hypersurface which lies between the bases A'B'C'D' and ABCD.

In another way, then, we can say that the tetrahedroidal-hyperprism A'B'C'D'-ABCD is make-up of 6 cells: 2 tetrahedrons and their interiors together with 4 lateral-prisms and their interiors. Each prism will have a lateral-face resting upon a lateral-face of each of the others, and each of the 4 faces of a base-tetrahedron resting upon 1 of the prisms.

The tetrahedroidal-hyperprism A'B'C'D'-ABCD is a right-hyperprism, its lateral-edges being perpendicular to the hyperplanes of the bases.

The visible-views of the tetrahedroidal-hyperprism in the graphic are the tetrahedron A'B'C'D' and its interior together with the prism A'B'C'-ABC and its interior. The interiors of the other 3 prisms will be hidden-views in the graphic except 2 faces on each 1 of the other 3 prisms will be a visible-view, that is, on each of these prisms we will have 1 lateral-face and 1 base as a visible-view: for example, the prism A'C'D'-ACD will have the lateral-face A'C'AC and base A'C'D' as visible-views in the graphic. The only visible-view in the base-tetrahedron ABCD, is the face ABC.

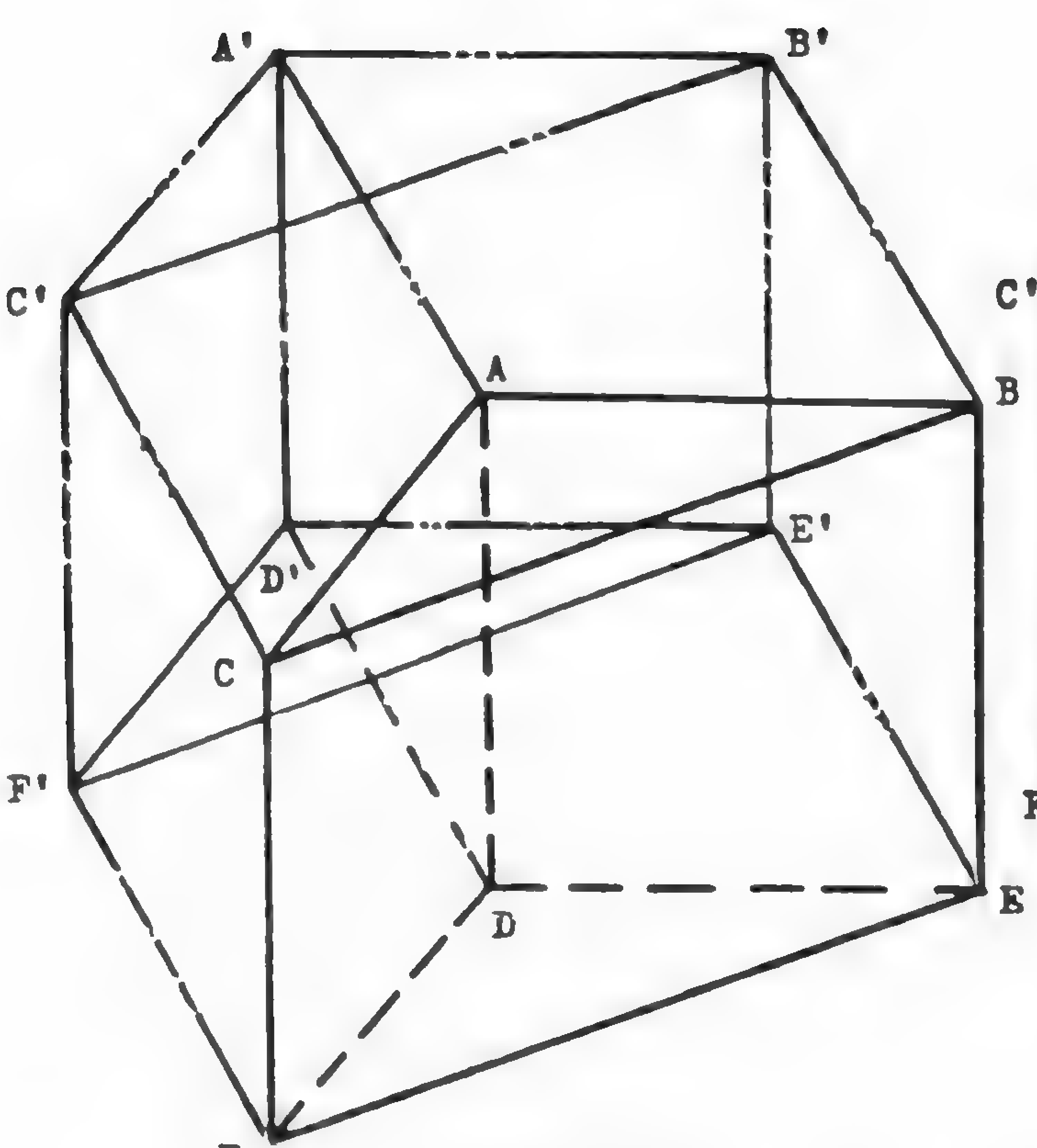


Fig. 80.

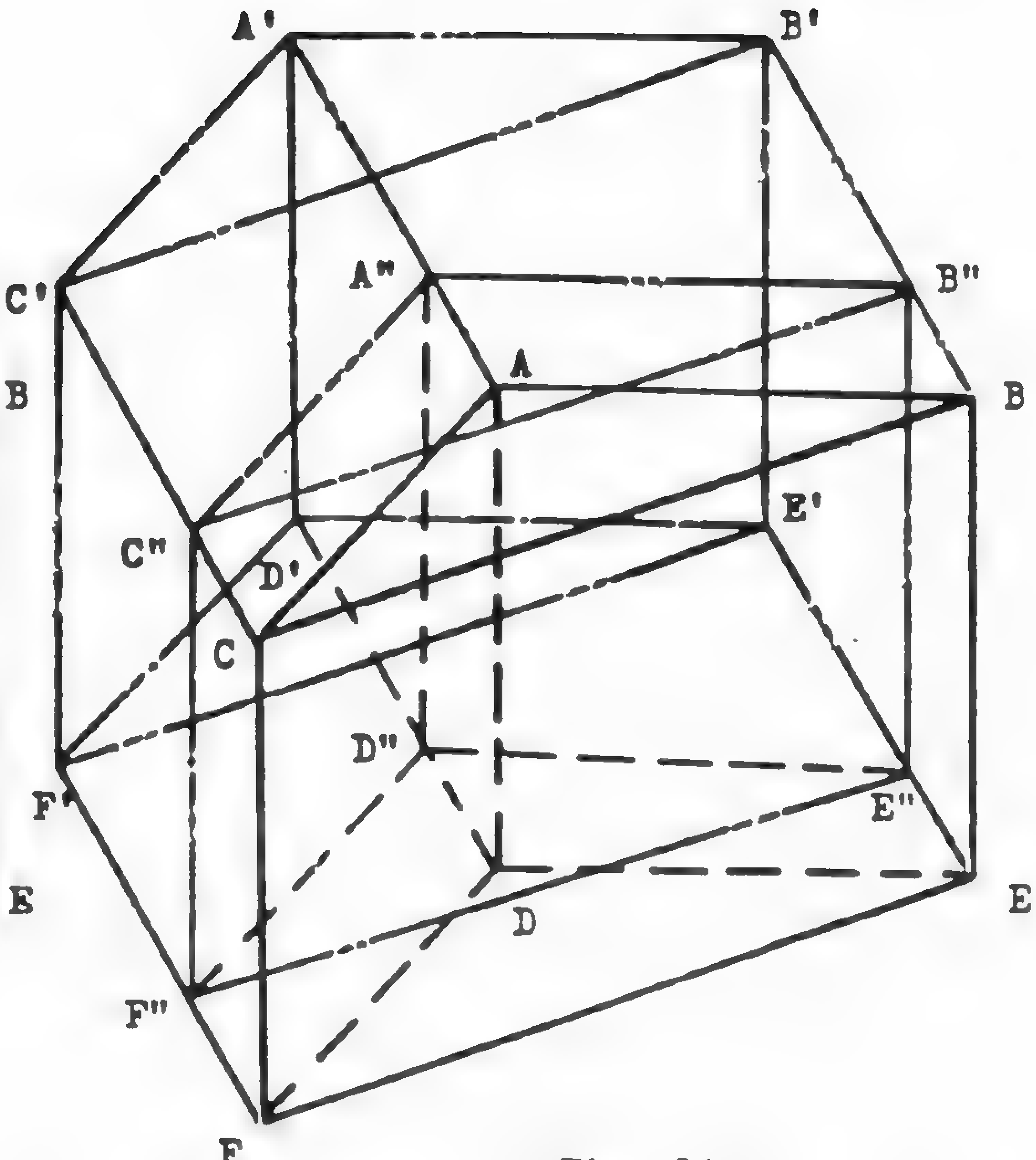


Fig. 81.

The tetrahedroidal-hyperprism A'B'C'D'-ABCD is the 4-space analogue of the triangular-prism B'C'D'-BCD in the solid-geometry (the hyperplane of the triangular-prism).

74. SPECIAL-FORMS OF HYPERPRISMS. HYPERPARALLELOPIPEDS. THE OCTAHEDROID. A hyperprism whose bases are the interiors of prisms can be regarded in 2-ways as a hyperprism of this kind; for the lateral-prisms corresponding to the ends of the bases are parallel (Art. 70, Th. 9) and congruent, and the remaining lateral-prisms are parallelopipeds, which can be regarded as having their bases on this 2nd pair of prisms and their lateral-edges those edges which belong to the 1st pair of prisms. This figure is a particular-case of a double-prism and will be studied in the next section (see Art. 80).

Fig. 80 represents a hyperprism whose bases are the interiors of triangular-prisms. The 2 lateral-prisms A'B'C'-ABC and D'E'F'-DEF corresponding to the ends of the bases A'B'C'-D'E'F' and ABC-DEF are parallel and congruent, and the remaining lateral-prisms A'B'D'E'-ABDE, A'C'D'F'-ACDF, and B'C'E'F'-BCEF are parallelopipeds, which can be regarded as having their bases on the 2nd pair of prisms A'B'C'-D'E'F' and ABC-DEF and their lateral-edges those edges which belong to the 1st pair of prisms A'B'C'-ABC and D'E'F'-DEF.

The hyperprism of Fig. 80 is a right-hyperprism. Its lateral-edges are all equal to unity, and the lateral-edges of the base-prisms are all equal to unity; its 3 parallelopipeds are cubes.

In Fig. 81, if we take a hyperplane perpendicular to the lateral-edge D'D at the point D" of a right-hyperprism with triangular-prisms for bases, then the hyperplane will intersect the hyperprism in a right-triangular-prism A"B"C"-D"E"F". The triangular-prism A"B"C"-D"E"F" is a right-section of the hyperprism.

A HYPERPARALLELOPIPED is a hyperprism whose bases are the interiors of parallelopipeds. In a hyperparallelopiped there are 4 pairs of opposite equal parallel parallelopiped whose interiors are the cells, and the interiors of any pair can be taken as bases. There are 4 sets of 8 parallel edges, each set joining the vertices of 2 opposite-cells, becoming the lateral-edges when the cells are taken as bases. The section of a hyperparallelopiped made by a hyperplane intersecting all 8 of the edges of a set will be a parallelopiped (Art. 72, Th. 4).

A DIAGONAL of a hyperparallelopiped is the interior of a segment formed by taking any 2 diagonally opposite vertices of the hyperparallelopiped.

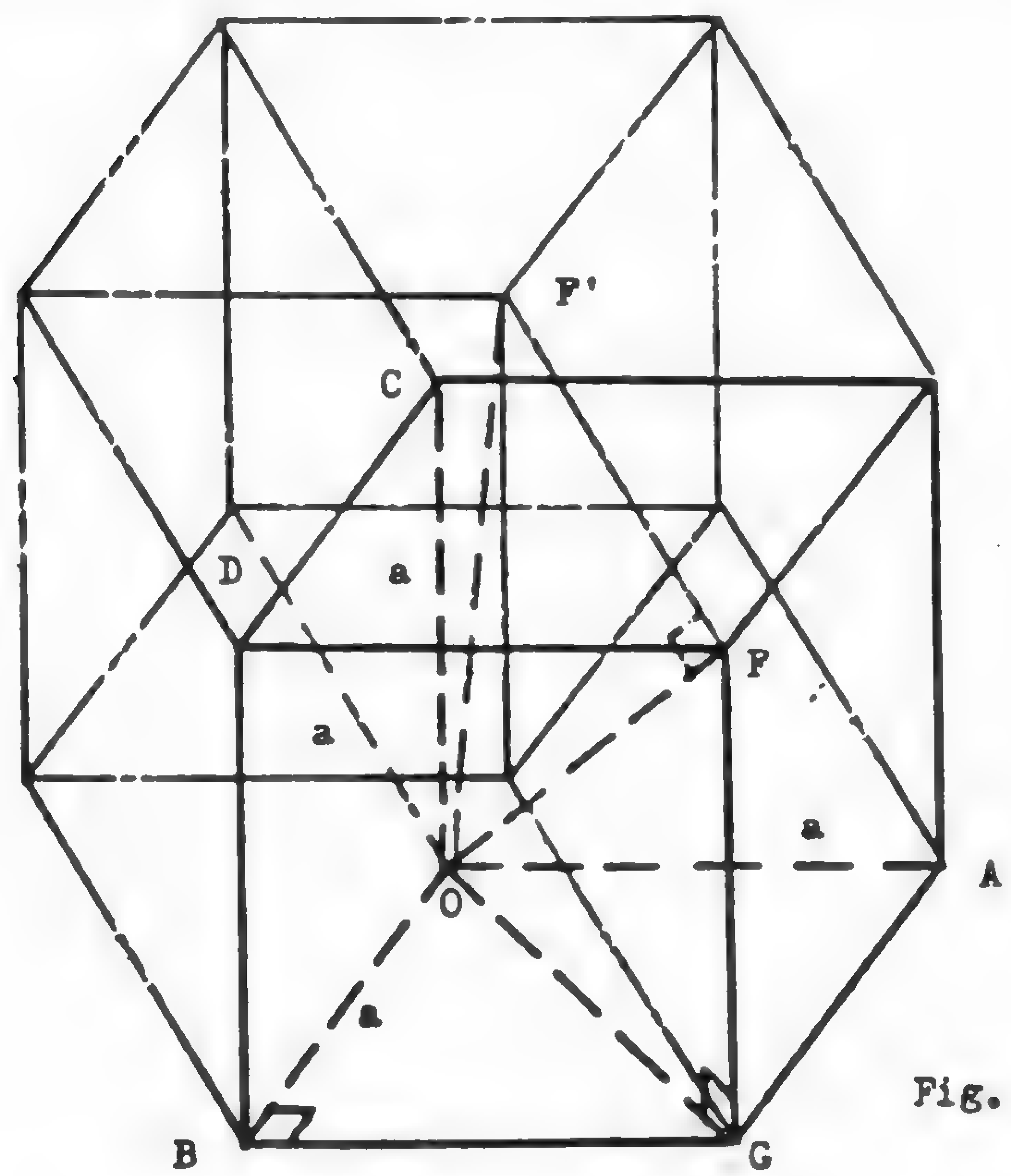
Theorem 1. The diagonals of a hyperparallelopiped bisect one another, all passing through a point which is a center of symmetry for the hyperparallelopiped.

Theorem 2. The square of the length of a diagonal of a rectangular-hyperparallelopiped is equal to the sum of the squares of the 4 dimensions.

The proof of the theorem follows from 3 successive applications of the Pythagorean theorem (See Fig. 82 for the special-case when the rectangular-hyperparallelopiped is a hypercube.

A HYPERCUBE is a rectangular-hyperparallelopiped whose base is the interior of a cube and whose altitude is equal to the edge of the cube. Its 4 dimensions are all equal. The hypercube is a REGULAR POLYHEDROID, being that there are 8 cells, it is called a REGULAR OCTAHEDROID; it has 8 equal cubical-cells, 24 equal faces each a common-face of 2 cubes, 32 equal edges, and 16 vertices. There are 4 axes lying in lines which also form a rectangular-system. The hypercube has also been called a TESSERACT.

Theorem 3. The diagonal of a hypercube is twice as long as the edge. (Fig. 82.)



(1) $OG^2 = a^2 + a^2 = 2a^2$

(2) $OF^2 = OG^2 + a^2 = 3a^2$

(3) $OF'^2 = OF^2 + a^2 = 4a^2$

(4) $OF' = 2a$

(4') $OF' = 2$, $a = 1$

Fig. 82.

In Fig. 82, take the vertex point G and construct the diagonal OG of the square OABG; take the vertex point F and construct the diagonal OF of the black-cube; take the vertex point F' and construct the diagonal OF' of the hypercube. Now take 3 successive applications of the Pythagorean theorem using the right-triangles OBG, OGF, and OFF'; the substitution of the value of OG^2 from the 1st equation into the 2nd, and the substitution of the value of OF^2 from the 2nd equation into the 3rd equation gives us the value of CF'^2; taking the square-root of both sides of equation (3) gives us equation (4), thus proving the theorem. Equation (4') results when a = 1.

III. DOUBLE-PRISMS

75. PLANO-PRISMATIC HYPERSURFACES. SECTIONS. We shall use the word LAYER to denote that portion of a hyperplane which lies between 2 parallel planes, and call the parallel planes the FACES OF THE LAYER.

A PLANO-PRISMATIC HYPERSURFACE consists of a finite number of parallel planes taken in a definite cyclical-order, and the layers which lie between consecutive planes of this order. The parallel planes are FACES and the layers are CELLS of the hypersurface. If α, β, γ ... are the faces in order, the cells can be described as the layers $\alpha\beta$, $\beta\gamma$, ... , and the hypersurface as the plano-prismatic hypersurface $\alpha\beta\gamma$... . The faces and all parallel planes within the layers are the ELEMENTS of the hypersurface, and are in cyclical-order.

The hypersurface is a SIMPLE PLANO-PRISMATIC HYPERSURFACE when no plane occurs twice as an element. It is CONVEX when also, the hyperplane of each cell contains no element except those of this cell and the 2 which are its faces. We shall consider only hypersurfaces which are simple and convex.

Fig. 83 can be taken as the graphic-representation of a plano-prismatic hypersurface, with the understanding that 'parallelograms' in the graphic represent planes and 'parallelopipeds' in the graphic represent hyperplanes of the layers. The 4 consecutive planes ABCD, EFGH, E'F'G'H', and A'B'C'D' are the faces of the layers. The 4 layers are ABCD-~~A'B'C'D'~~ EFGH, ~~A'B'C'D'~~ EFGH-E'F'G'H', E'F'G'H'-A'B'C'D', and A'B'C'D'-ABCD.

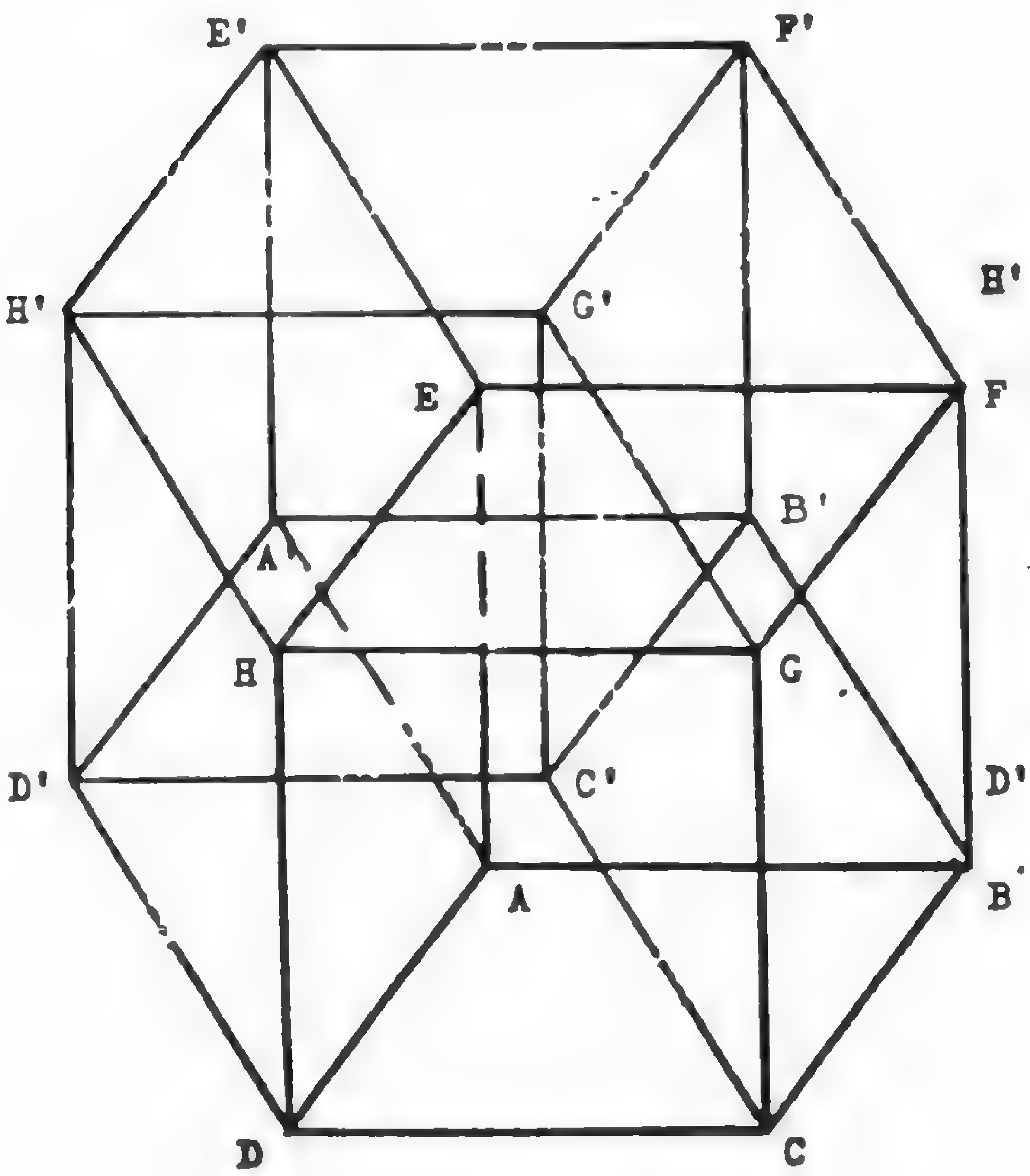


Fig. 83.

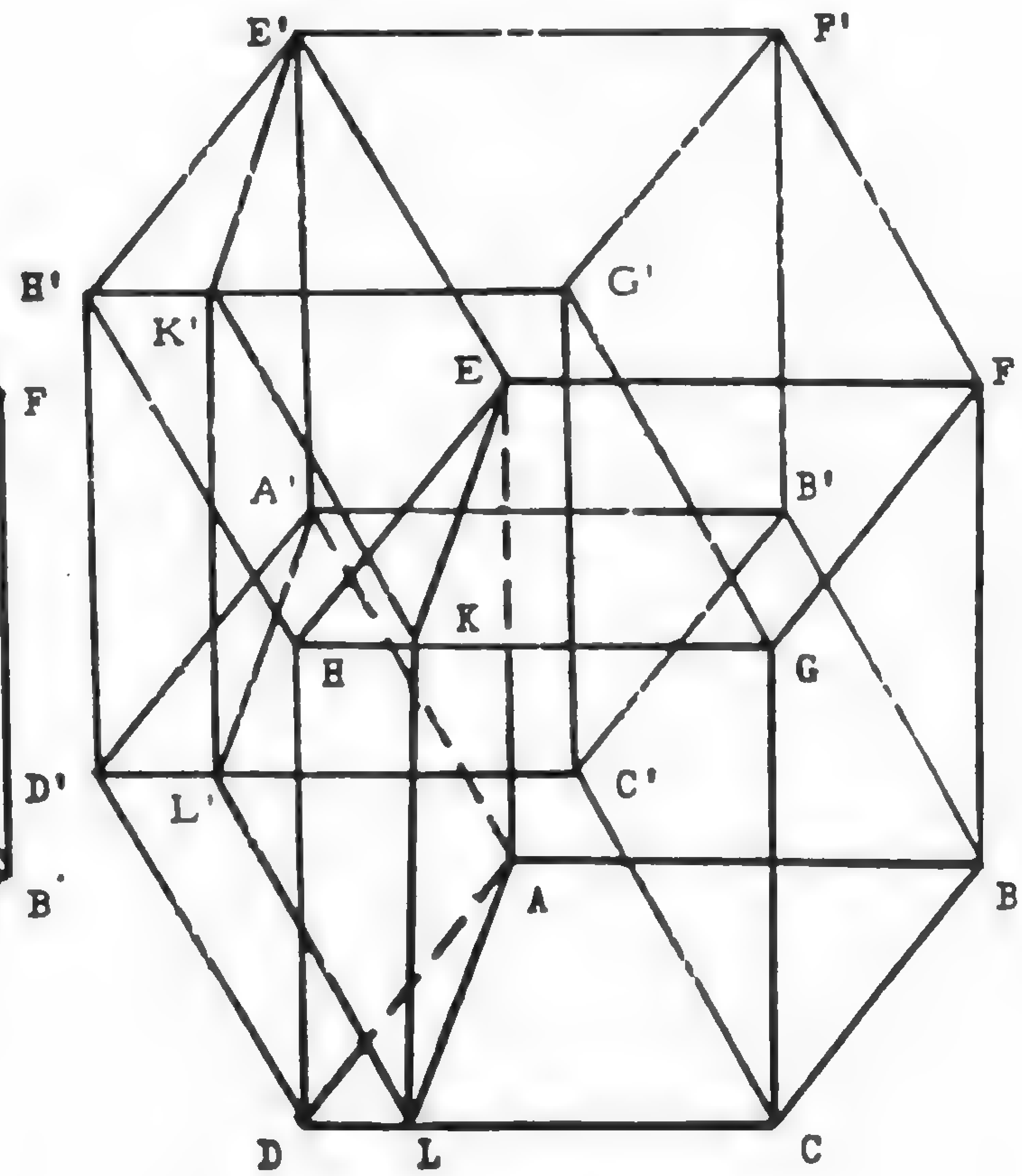


Fig. 84.

Theorem 1. A hyperplane containing an element of a plano-prismatic hypersurface or any parallel plane will intersect the hypersurface in elements if at all (Art. 70, Th. 1).

Theorem 2. A hyperplane intersecting but not containing an element of a plano-prismatic hypersurface will intersect the hypersurface in a prismatic-surface. (Fig. 84.)

For the hyperplane intersects all the elements in parallel lines (Art. 68, Th. 4), and so the cells in strips (see Art. 72), which are the faces of a prismatic-surface.

In Fig. 84, a hyperplane intersecting the face-elements of the plano-prismatic hypersurface in the 4 parallel lines AL, EK, E'K', and A'L' will intersect the hypersurface in a prismatic-surface LKK'L'-AEE'A'. The faces of the prismatic-surface lie in the cells of the hypersurface.

Theorem 3. 2 parallel prismatic-sections of a plano-prismatic hypersurface are congruent.

76. DIRECTING-POLYGONS OF A PLANO-PRISMATIC HYPERSURFACE. INTERIOR OF THE HYPERSURFACE.

Theorem 1. A plane which contains a line parallel to the elements of a plano-prismatic hypersurface, but is not itself parallel to them, will intersect in a line every element which it intersects if at all.

Theorem 2. A plane which does not contain a line parallel to the elements of a plano-prismatic hypersurface will intersect every element in a point, and will intersect the hypersurface, the latter being convex, in a convex-polygon.

For the plane will intersect each element in a point, by Th. 5 of Art. 68, and the faces of the hypersurface in points which are the vertices of a polygon. The polygon is a simple convex-polygon, since no 2 elements can intersect the same plane in the same point.

In Fig. 84, a plane intersecting the face-elements of a plano-prismatic hypersurface in points L, K, K', and L', will intersect the hypersurface in a rectangle LKK'L', the points L, K, K', and L' being the vertices of this rectangle.

The polygon in which a plane containing no line parallel to the elements intersects the hypersurface can be called a DIRECTING-POLYGON, and the hypersurface can be described as consisting of a system of parallel planes passing through the points of a given polygon and intersecting the plane of the polygon only in these points. The polygon GOES AROUND the hypersurface.

In Fig. 84, the parallelogram LKK'L' is called a directing-parallelogram of the hypersurface, which goes around the hypersurface. Another directing-parallelogram of the hypersurface is AEE'A'. All the directing-parallelograms of the hypersurface are congruent to one another.

Theorem 3. Through any line which is not parallel to the elements of a plano-prismatic hypersurface can be passed planes intersecting the hypersurface in directing-polygons.

Theorem 4. 2 parallel directing-polygons of a plano-prismatic hypersurface are congruent, and any 2 homologous points lie in 1 of the elements.

In Fig. 83, if we take the 2 parallel directing-parallelograms AEE'A' and DHH'D', then they are congruent. Their interiors are the bases a prism AEE'A'-DHH'D' whose lateral-surface is cut-out from the prismatic-surface in which their hyperplane intersects the hypersurface.

Theorem 5. If a plane-prismatic hypersurface has a parallelogram for directing-polygon, it will have 2 pairs of equal opposite-cells (layers of the same width) lying in parallel hyperplanes, and all its directing-polygons will be parallelograms.(see Art. 70, Ths. 8 and 9). (Fig. 83.)

The 2 cells EFGH-ABCD and E'F'H'G'-A'B'C'D' lie in parallel hyperplanes. They form a pair of opposite-cells of the hypersurface and are layers of the same width, since AEE'A' is a directing-parallelogram of the plano-prismatic hypersurface, with AE = A'E' and EE' = AA'. Like results occur also for the other 2 remaining cells of the hypersurface.

The INTERIOR OF A PLANO-PRISMATIC HYPERSURFACE consists of those planes which correspond to the interior of the directing-polygon. The interior of any segment whose points are points of the hypersurface will lie entirely in the interior of the hypersurface unless it lies entirely in the hypersurface itself, and a ½-line drawn from a point of the interior and not parallel to the elements will intersect the hypersurface in 1 and only 1 point. Also that portion of a plane between 2 parallel lines of the hypersurface (a strip, Art. 68), or a layer between any 2 elements, lies entirely in the interior unless it lies in the hypersurface itself.

Theorem 6. When the layer between 2 elements of a plano-prismatic hypersurface lies entirely in the interior of the hypersurface, it separates the rest of the interior into 2 portions lying on opposite-sides of the hyperplane, and with its faces and each of the 2 parts into which they separate the rest of the hypersurface it forms a convex plano-prismatic hypersurface.

Corollary. By taking the diagonal-layers which have in common 1 of the lateral-faces of the hypersurface we can form a set of triangular plano-prismatic hypersurfaces, their interiors, together with the diagonal-layers, making up the interior of the given hypersurface.

76. RIGHT DIRECTING-POLYGONS. AXIS-PLANES. A directing-polygon whose plane is absolutely-perpendicular to the planes of the elements of a plano-prismatic hypersurface is called a RIGHT DIRECTING-POLYGON.

Theorem 1. The projection of any directing-polygon upon the plane of a right directing -polygon is the right directing-polygon itself (see Art. 18).

Theorem 2. A plane isocline to the plane of a right directing-polygon, but not parallel to the elements, intersects the hypersurface in a polygon similar to the right directing-polygon; and, conversely, any directing-polygon similar to a right directing-polygon lies in a plane which is isocline to the plane of the latter (see Art. 71, Ths. 1 and 2). The right directing-polygon is the minimum of all these similar polygons.

Theorem 3. If any directing-polygon of a plano-prismatic hypersurface has a center of symmetry, the plane through this point parallel to the elements is an axis-plane of symmetry, meeting the plane of every directing-polygon in a point which is a center of symmetry of this polygon. Each point of the plane is, in fact, a center of symmetry for the entire hypersurface, every line of the plane is a line of symmetry, and the plane as-a-whole is a plane of symmetry.

For the plane lies mid-way between 2 planes in which any hyperplane containing it intersects the hypersurface, and any line intersecting it but not lying in it determines with it such a hyperplane.

77. INTERSECTION OF 2 PLANO-PRISMATIC HYPERSURFACES. THE 2 SETS OF PRISMS. When the elements of a plano-prismatic hypersurface intersect the elements of a 2nd plano-prismatic hypersurface only in points, the intersection of the 2 hypersurfaces consists of the lateral-surfaces of a set of prisms joined together in succession by their bases, together with the polygons whose interiors are these bases. In another way, also, the same intersection consists of the lateral-surfaces of a set of prisms joined together by their bases, together with the polygons whose interiors are these bases.

In fact, the faces of the 1st hypersurface are parallel planes intersecting the 2nd hypersurface in a set of equal parallel directing-polygons of the latter, and the cells of the 1st are layers, each layer intersecting the 2nd in the lateral-surface of a prism whose bases are interiors of 2 of these directing-polygons (see Art. 76, Th. 4).

In the same way the faces of the 2nd hypersurface are parallel planes intersecting the 1st hypersurface in a set of parallel directing-polygons, and each cell of the 2nd intersects the 1st in the lateral-surface of a prism whose bases are the interiors of 2 of these directing-polygons. The entire intersection consists in 2-ways of the lateral-surfaces of a set of prisms joined in succession by their bases, together with the polygons whose interiors are these bases.

The lateral-faces of any prism of the 1st set are the interiors of parallelograms, and are the intersections of 1 particular cell of the 1st hypersurface with the different cells of the 2nd. A set of corresponding faces of these prisms, 1 from each prism, are, then, the intersections of the different cells of the 1st hypersurface with 1 particular cell of the 2nd. The faces of any particular prism of either set form a set of corresponding faces of the different prisms of the other set, and every lateral-face of a prism of one set is a lateral-face of some prism of the other set. The lateral-edges of the prisms of one set are the sides of the bases of the prisms of the other set. The interiors and the bases of the prisms of the 1st set lie in the 1st hypersurface and in the interior of the 2nd, and the interiors and bases of the prisms of the 2nd set lie in the 2nd hypersurface and in the interior of the 1st. The 1st set of prisms goes around the 1st hypersurface, while any base or interior of a cross-section of any of these prisms is a piece cut-out of an element of the 1st hypersurface. In the same way the 2nd set of prisms goes around the 2nd hypersurface.

A DOUBLE-PRISM consists of the intersections of 2 plano-prismatic hypersurfaces whose elements intersect only in points, together with all that portion of each hypersurface

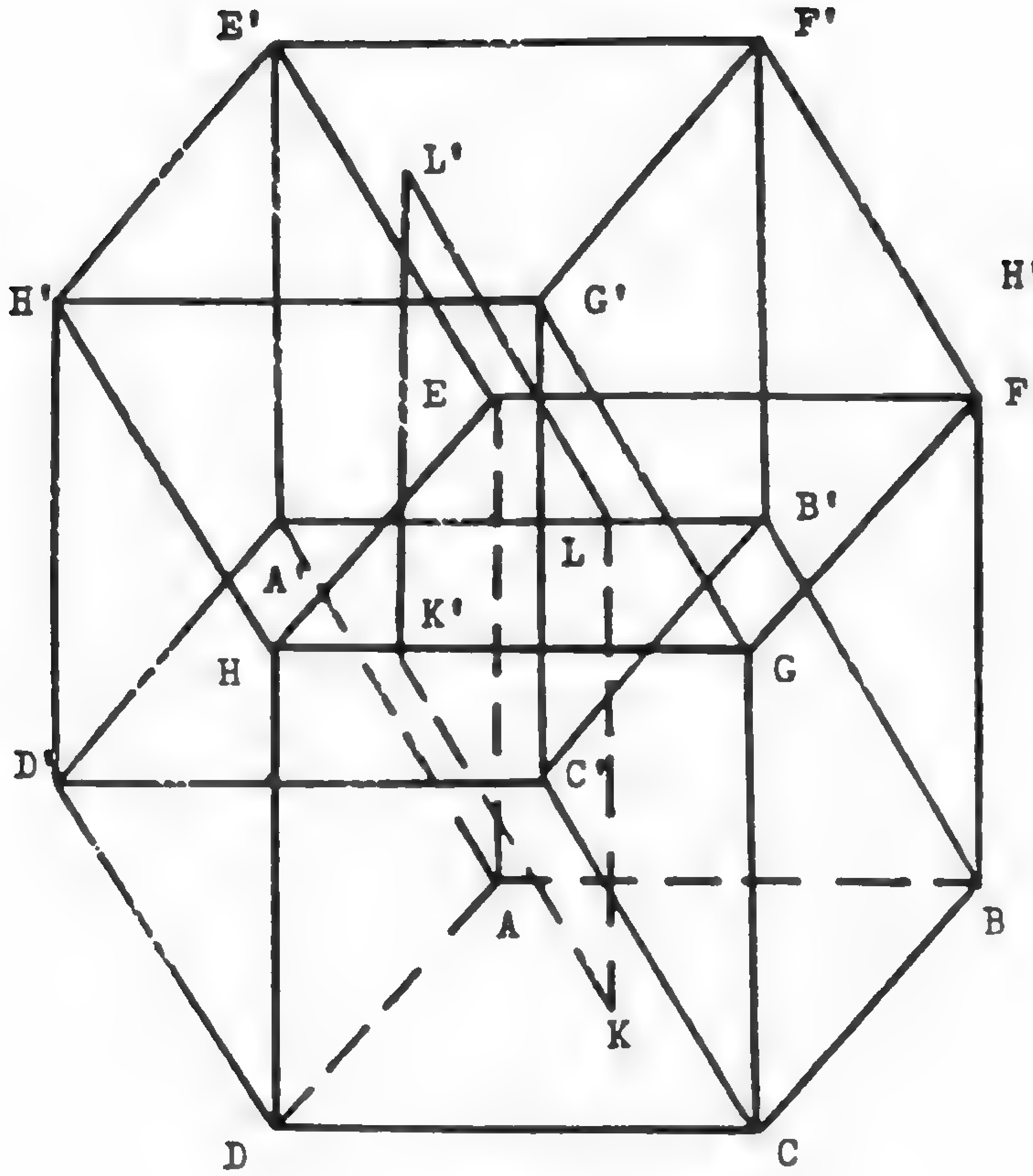


Fig. 85.

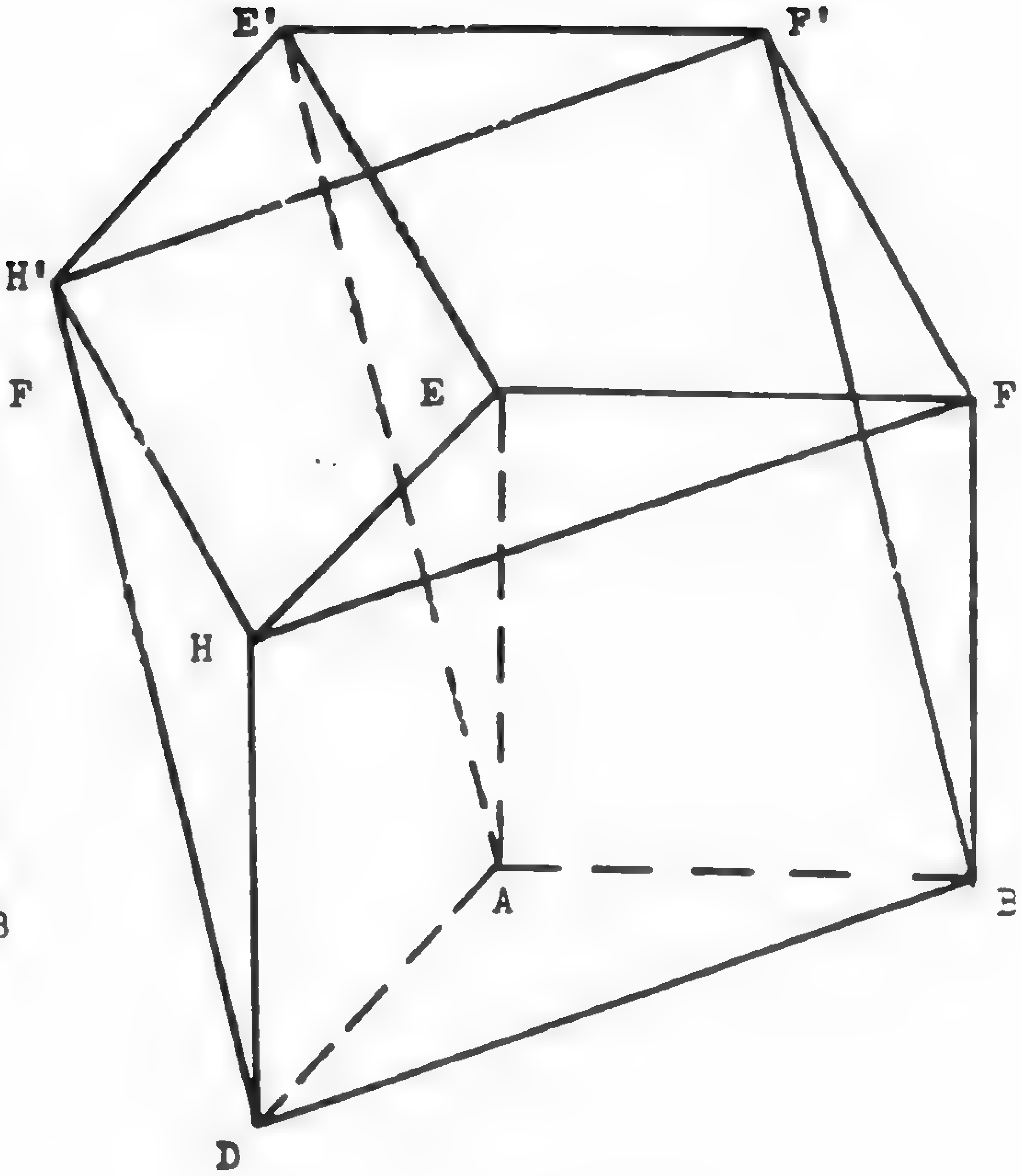


Fig. 86.

which lies in the interior of the other; that is, it consists of both sets described above and their interiors.

When the elements of one hypersurface are absolutely-perpendicular to the elements of the other the double-prism is a RIGHT DOUBLE-PRISM. When also the prisms of the 2 sets are regular the double-prism is REGULAR.

If we take the hypercube given in chapter I, then, as described above, the hypercube will be formed from the intersection of 2 plano-prismatic hypersurfaces having squares as directing-polygons, with the planes of these squares absolutely-perpendicular to each other.

The 4 cubes that go around the lateral-faces of the black-cube can be taken as the 1st set of prisms lying in the 1st hypersurface, and the 4 cubes that go around the lateral-faces of the cube OBHC-C'B'H'C' can be taken as the 2nd set of prisms lying in the 2nd hypersurface.

The hypercube is a special-case of a double-prism, and since the elements of the 2 hypersurfaces containing the hypercube are absolutely-perpendicular to each other, the hypercube is a right double-prism. The hypercube is a regular double-prism, since both sets of prisms are cubes (regular-prisms).

78. INTERIOR OF A DOUBLE-PRISM. THE DIRECTING-POLYGONS. The INTERIOR OF A DOUBLE-PRISM consists of the points which are common to the interiors of its 2 hypersurfaces. A plane lying in the interior of one of the hypersurfaces parallel to its elements intersects the other in a directing-polygon whose interior belongs to the interiors of both, and so to the interior of the double-prism. The vertices of this polygon are a set of corresponding points of the bases, and the sides lie in the interiors of the prisms of the set which goes around the former hypersurface.

In Fig. 85, the plane of the parallelogram KLL'K' lies in the interior of the hypersurface containing the set of prisms that go around the prism EFGH-ABCD as well as being parallel to the elements of this hypersurface. The plane of KLL'K' intersects the other hypersurface in a directing-polygon which is a parallelogram, the interior of which belongs to the interiors of both hypersurfaces, and so to the double-prism. The vertices K, L, L', and K'

are a set of corresponding points of the bases, and the sides KL, LL', L'K', and K'K lie in the interiors of the prisms which go around the former hypersurface.

Theorem 1. Any plane through a point of the interior of a convex double-prism intersects the double-prism in a convex-polygon.

Corollary. The interior of any segment whose points are points of a double-prism will lie entirely in the interior of the double-prism unless it lies entirely in the interior of the double-prism, and a $\frac{1}{2}$-line drawn from a point of the interior will intersect the double-prism in 1 and only 1 point.

In a double-prism the directing-polygons of each hypersurface whose planes are elements of the other is called THE DIRECTING-POLYGONS OF THE DOUBLE-PRISM. Any 2 polygons intersecting in a single point and lying in the planes which have only this point in common can be taken each as a directing-polygon with the plane of the other as element of a plano-prismatic hypersurface, and so the 2 together as the directing-polygons of a double-prism.

In another way, we can say that the surface of intersection of the 2 hypersurfaces, the common lateral-surfaces of the 2 sets of prisms as described in the preceding article, is generated by moving 1 of these polygons (generating-polygon) keep parallel to itself around the other (directing-polygon). Each point of the polygon moves along the prisms of one set and around 1 of the prisms of the other set. The interior of the polygon generates the interiors of the prisms along which it moves. The interiors of the other prisms will be generated by the interior of the other polygon mooving in the same way around the 1st. The surface of intersection is covered with the polygons of each set, the 2 sets forming on it a net. (In Fig. 85, both of the polygons ABCD and AE'A'A are directing and generating-polygons of the double-prism.)

79. CUTTING A DOUBLE-PRISM SO AS TO FORM 2 DOUBLE-PRISMS. DOUBLY TRIANGULAR-PRISMS.

Theorem. When a double-prism is cut by a hyperplane passing through points of the interior and containing elements of one hypersurface, the intersection is a prism, and the rest of it is separated into 2 portions, which, each combined with the prism and its interior, form 2 double-prisms whose interiors, with that of the prism, make-up the whole interior.

Corollary. By cutting a double-prism diagonally we can form double-prisms in which the prisms of one set are triangular, so that those of the other set are 3 in number; and then , cutting these in another way diagonally, we can form double-prisms in which the prisms of both sets are triangular, the interiors of all these double-prisms together with the interiors of the prisms of intersection making-up the whole interior.

A double-prism in which the prisms of both sets are triangular is a DOUBLY-TRIANGULAR DOUBLE-PRISM, or simply a DOUBLY-TRIANGULAR-PRISM. Such a double-prism is formed when any 3 hyperplanes intersecting by 2's in 3 parallel planes are cut by 3 other hyperplanes which intersect by 2's in 3 parallel planes, any plane of one set intersecting any plane of the other set only in a single point.

Fig. 86 is the graphic-representation of a doubly-triangular double-prism. Any plane of a triangle of one set intersecting any plane of a triangle of the other set only in a single point.

80. HYPERPRISMS WITH PRISM-BASES AS DOUBLE-PRISMS. HYPERPARALLELOPIPEDS. CENTER OF SYMMETRY. A hyperprism whose bases are the interiors of prisms is a double-prism, the 2 prisms of the bases and the 2 lateral-prisms corresponding to the ends forming 1 of the 2 sets of prisms of the double-prism, while the prisms of the other set are parallelopipeds (see Art. 78).
Conversely, a double-prism in which the prisms of one set are parallelopipeds (and therefore the prisms of the other set are 4 in number, 2 pairs of opposites) can be regarded in 2-ways as a hyperprism, the bases in each case being the interiors of a pair of opposite-prisms of the 2nd set.

Fig. 80 is a hyperprism with prism-bases, and therefore, a double-prism. The bases are the interiors of the 2 triangular-prisms A'B'C'-D'E'F' and ABC-DEF. The 2 lateral-prisms are A'B'C'-ABC and D'E'F'-DEF. These 4 triangular-prisms form one of the sets of prisms

of the double-prism, and the other set of prisms are the 3 parallelopipeds A'B'D'E'-ABDE, A'C'D'F'-ACDF, and B'C'L'F'-BCEF.
The double-prism of Fig. 80 can also be regarded in 2-ways as a hyperprism as follows: the pair of opposite-prisms A'B'C'-D'E'F' and ABC-DEF (lying in parallel hyperplanes) forming the bases of the hyperprism; the pair of opposite-prisms A'B'C'-ABC and D'E'F'-DEF forming the bases of the hyperprism in the other case.

Theorem 1. When the prisms of both sets in a double-prism are parallelopipeds, or, what is the same thing, when both of directing-polygons are parallelograms, the figure is a hyperparallelopiped. Indeed, the hyperparallelopiped can be regarded in 3-ways as a double-prism, the parallelopipeds of 2 pairs of opposite-cells forming one of the sets of prisms and the other 4 parallelopipeds the other set.

Theorem 2. When the 2 hypersurfaces of a double-prism have axis-planes of symmetry, the point of intersection of these planes is a center of symmetry of the double-prism; and any hyperplane through this point intersects the double-prism in a polyhedron which divides the rest of the double-prism into 2 congruent parts (Art. 49, Th. 2). (Fig. 87.)

In Fig. 87, the 2 hypersurfaces of the double-prism have as axis-planes the planes of the directing-parallelograms KLL'K' and MNN'M', which intersect in a point O, the center of symmetry of the double-prism. A diagonal-hyperplane will pass through O and intersect the double-prism in a diagonal-prism (parallelopiped) F'H'B'D'- FHBD which divides the rest of the double-prism into 2 congruent parts.
When the axis-planes of the 2 hypersurfaces of the double-prism are absolutely-perpendicular to each other at O, then if the double-prism is a rectangular-hyperparallelopiped, we can construct a set of rectangular-axes at O. 2 axes will lie in 1 axis-plane, and the other 2 axes will lie in the other axis-plane.

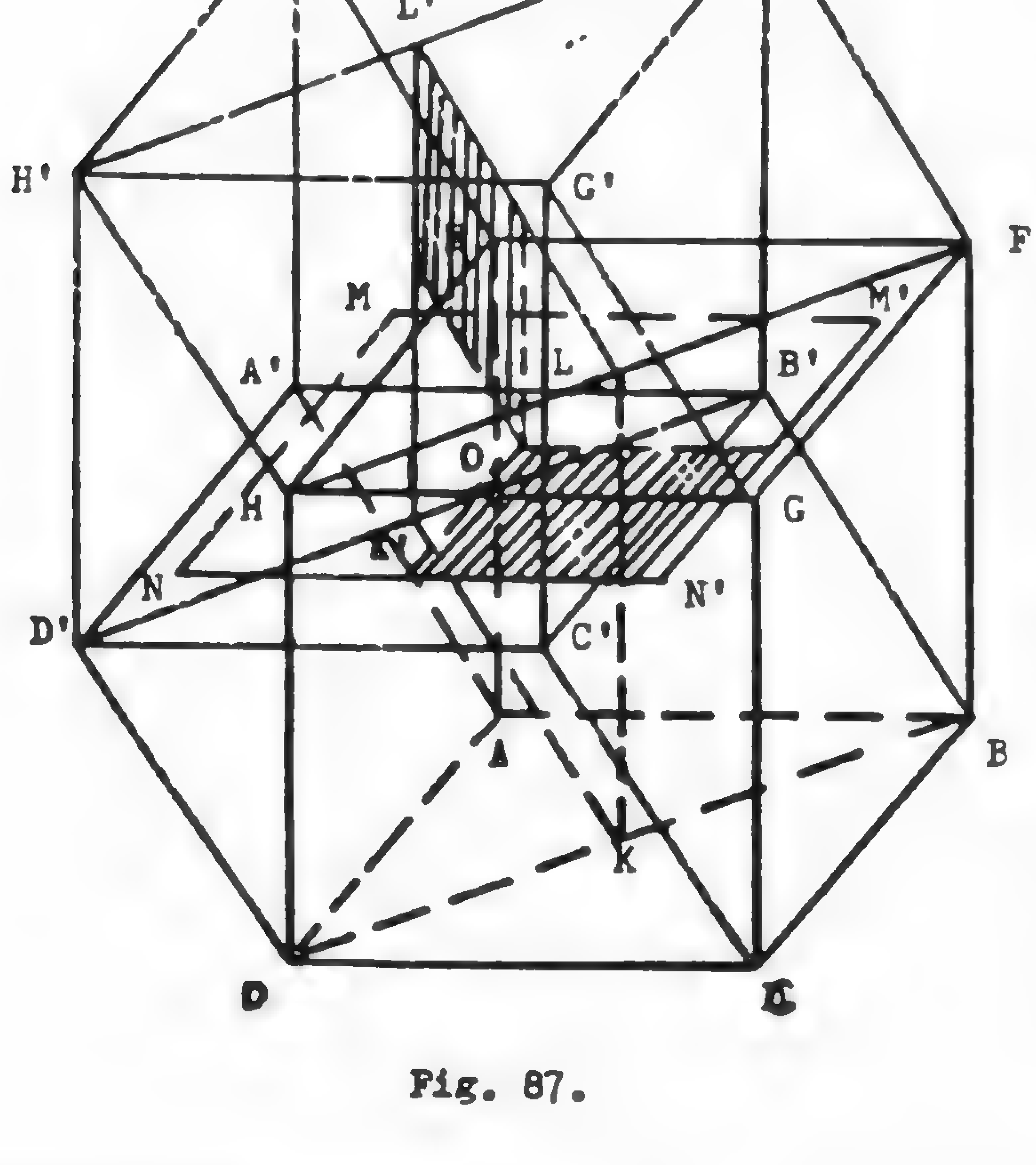


Fig. 87.

81. HYPERCYLINDRICAL-HYPERSURFACES. INTERIORS. SECTIONS. A HYPERCYLINDRICAL-HYPERSURFACE consists of a system of parallel lines passing through the points of a hyperplane-surface, but not lying in the hyperplane of this surface. The surface is called the DIRECTING-SURFACE, and the parallel lines are the ELEMENTS.

We shall consider only those cases in which the directing-surface is a surface of solid-geometry, a plane or a sphere, or a conical or cylindrical-surface with directing-circle, or a part or combination of parts of such surfaces. A prismoidal-hypersurface is a particular case of a hypercylindrical-hypersurface.

Many of the properties of the hypersurface correspond to the properties of the directing-surface. The hypersurface has an interior when the directing-surface has an interior, the INTERIOR OF THE HYPERSURFACE consisting of the lines which pass through the points of the interior of the directing-surface and are parallel to the elements.

Sections of the hypercylindrical-hypersurface are like those of the prismoidal-hypersurface: a hyperplane passing through a point of the interior and parallel to the elements intersects the hypercylindrical-hypersurface in a cylindrical-surface, and a hyperplane which is not parallel to the elements intersects the hypersurface in a surface, or at least in a system of points, which can serve as a directing-surface. A RIGHT-SECTION is a directing-surface whose hyperplane is perpendicular to the elements.

Theorem. Sections of a hypercylindrical-hypersurface made by parallel hyperplanes not parallel to the elements are congruent.

82. SPECIAL-FORMS OF HYPERCYLINDERS. SPHERICAL-HYPERCYLINDERS. A HYPERCYLINDER consists of that portion of a closed hypercylindrical-hypersurface which lies between 2 parallel directing-surfaces, together with the directing-surfaces themselves and their interiors.

The interiors of the directing-surfaces are the BASES, and that portion of the hypercylindrical-hypersurface which lies between the directing-surfaces is the LATERAL-HYPERSURFACE of the hypercylinder. The INTERIOR OF THE HYPERCYLINDER consists of that portion of the interior of the hypercylindrical-hypersurface which lies between the bases.

A SPHERICAL-HYPERCYLINDER is one whose bases are the interiors of spheres. The AXIS OF A SPHERICAL-HYPERCYLINDER is the interior of a segment whose points are the centers of the bases.

Fig. 88 is the graphic-representation of a spherical-hypercylinder which can be denoted by S'S. Its bases are the interiors of the 2 spheres S' and S. The lateral-hypersurface of S'S is that portion of the closed spherical-hypercylindrical-hypersurface which lies between the directing-spheres S' and S. The interior of S'S consists of that portion of the interior of the spherical-hypercylindrical-hypersurface which lies between the bases. The axis of S'S is the interior of the segment O'O whose points O' and O are the centers of the bases.

The visible and hidden-views in the graphic of the spherical-hypercylinder S'S are analogous to that of a circular-cylinder in the solid-geometry. The red-sphere S' and its interior will be a visible-view in the graphic, and that portion of the lateral-hypersurface lying on one side of the hyperplane of the circular-cylinder (A'B'C')-(ABC), with the point F being that side, will be a visible-view in the graphic, or what is the same thing, that portion of the lateral-hypersurface of S'S lying between the ½-spheres S'_F, and S_F will be a visible-view in the graphic and a ½-lateral-hypersurface of S'S. The other ½-lateral-hypersurface of S'S lying between the ½-spheres S'_E, and S_E will be a hidden-view in the graphic. The interior of S will be a hidden-view, and the ½-spherical-surface S_F will be the only visible-view of S.

The circular-cylinder (A'B'C')-(ABC) and its interior divides the spherical-hypercylinder S'S into 2 congruent parts of ½-spherical-hypercylinders, and its lateral-surface lies in the lateral-hypersurface of S'S and is a visible-view in the graphic.

Theorem 1. When a spherical-hypercylinder is cut by a hyperplane which passes through a point of the interior and is parallel to the elements, the intersection is a circular-cylinder.

This theorem is the analogue to a theorem in the solid-geometry for cylinders. The student should compare the visual-graphics between the hypercylinder and cylinder.

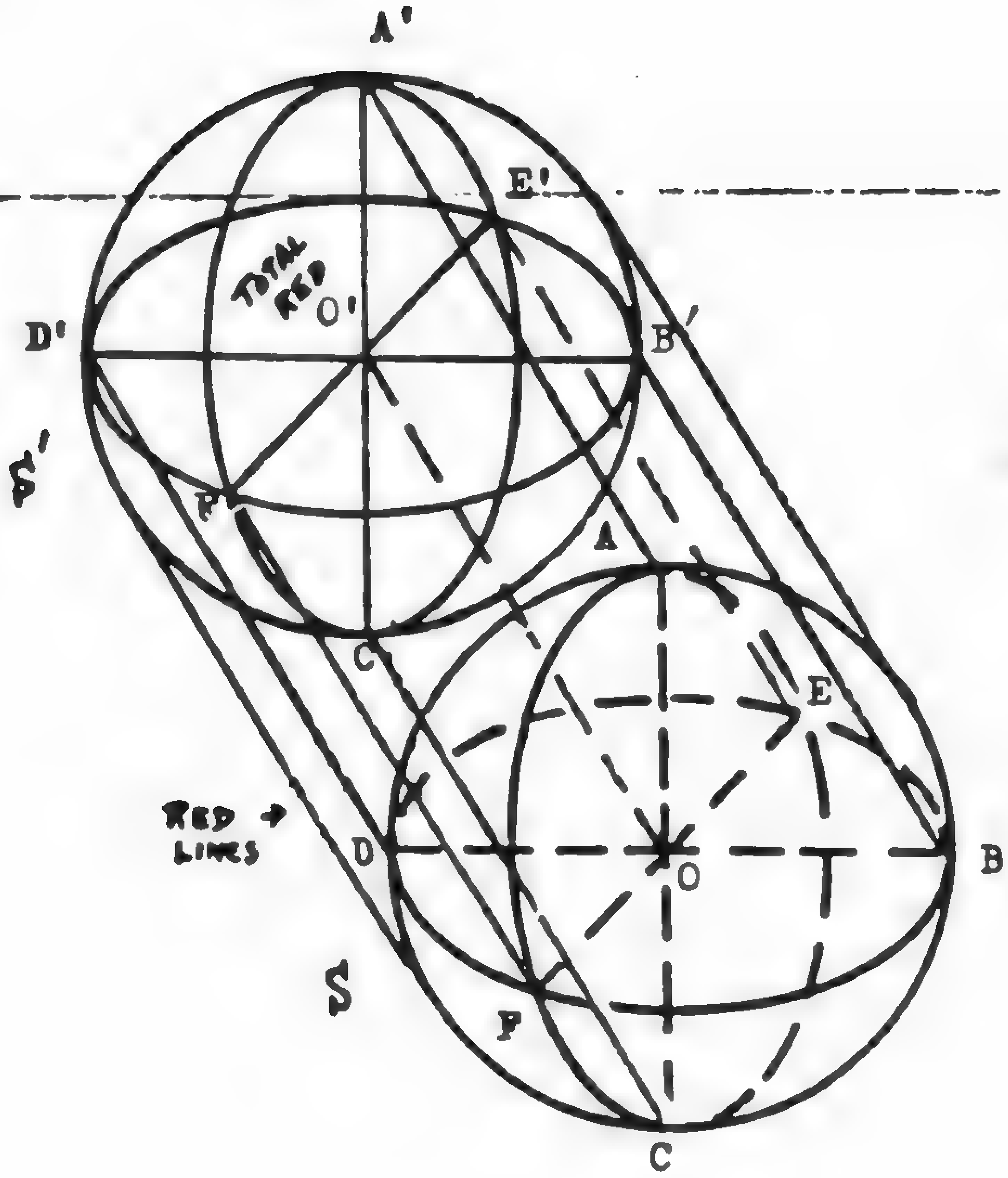


Fig. 88.

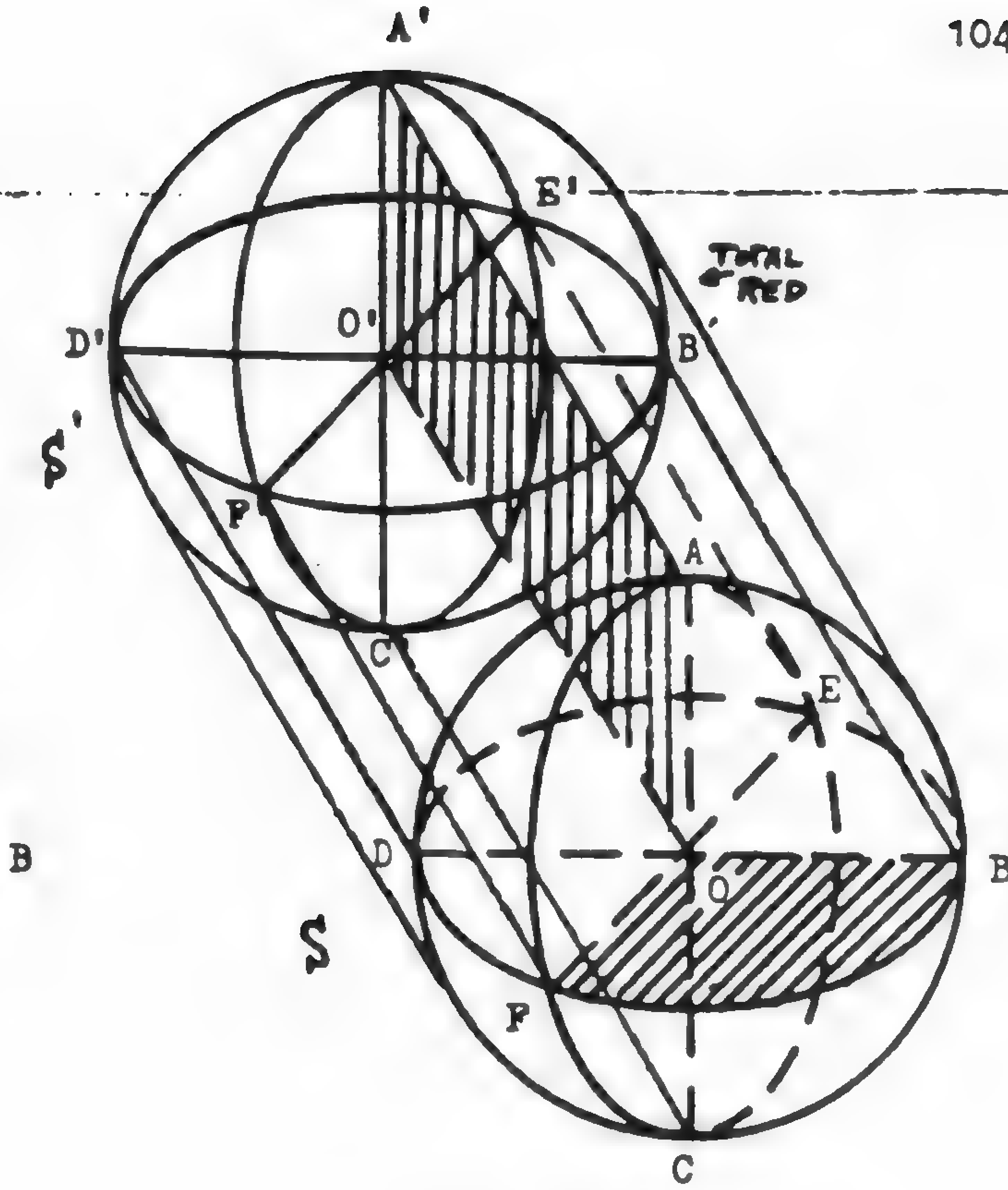


Fig. 89.

Theorem 2. When a rectangle takes all positions possible with 1 side fixed, the vertices and the points of the other 3 sides make-up a right spherical-hypercylinder. The fixed-side is the axis, the opposite-side is an element, and the other 2 sides are the radii of the bases. (Fig. 89.)

In Fig. 89, when a rectangle OAA'O' takes all positions possible with the side OO' fixed, the vertices and points of the 3 sides OA, AA', and A'O' make-up a right spherical-hypercylinder S'S. The opposite-side AA' (of OO') is an element, and the sides OA and O'A' are radii of the bases S and S' respectively.

The vertices A' and A of the rectangle OAA'O' will generate the spheres S' and S respectively; the sides O'A' and OA will generate the interiors of the spheres S' and S respectively; the side A'A will generate the lateral-hypersurface of S'S; and the interior of the rectangle OAA'O' will generate the interior of S'S.

Theorem 3. If we pass a plane through the axis of a cylinder of revolution and rotate around this plane that portion of the cylinder which lies to one side of it, we shall form all of a right spherical-hypercylinder except that portion which is the intersection of the cylinder by the plane. (Fig. 89.)

In Fig. 89, if we take the plane of the rectangle A'ACC' which passes through the axis OO' of a cylinder of revolution (A'D'C')-(ADC), then if we rotate around the plane of the rectangle A'ACC' the ½-cylinder (A'D'C')-(ADC) lying on one side of it, we shall form all of a right spherical-hypercylinder S'S except the rectangle A'ACC', which is the intersection of the cylinder (A'D'C')-(ADC) by the plane of this rectangle.

A hypercylinder whose bases are the interiors of cylinders can be regarded in 2-ways as a hypercylinder of this kind; for there are 2 lateral-cylinders corresponding to the ends of the bases, and these can be taken as the bases and the given bases as parts of the lateral-hypersurface.

The lateral-cylinders are congruent, and lie in parallel hyperplanes, with the elements of one parallel to the other and the planes of the bases of one parallel to the corresponding planes of the other. Further, those elements of the hypercylinder whose lines intersect any element of one of its bases lie in the interior of a parallelogram which bears the same relation to both pairs of cylinders. This figure is a particular case of a prism-cylinder, and will be studied in the next section (see Art. 85).

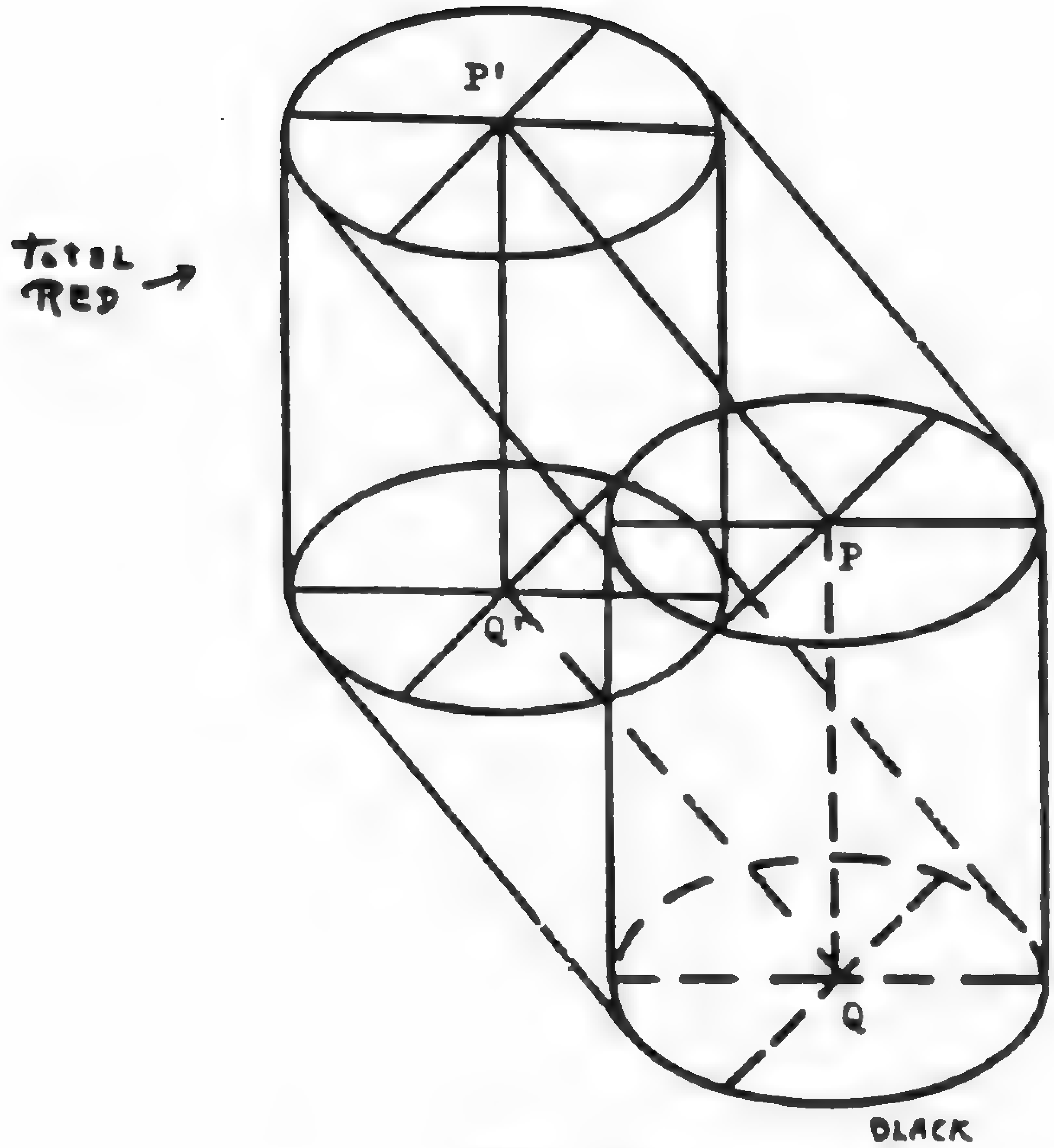


Fig. 90.

Fig. 90 is the graphic-representation of a hypercylinder whose bases are the interiors of cylinders and may be denoted by P'Q'PQ. We will write CYLINDER PQ to denote a cylinder of P'Q'PQ. We will write AXIS PQ to denote the axis of a cylinder of the hypercylinder.
The hypercylinder P'Q'PQ can be regarded in 2-ways as a hypercylinder of this kind; for the lateral-cylinders P'P and Q'Q corresponding to the ends of the bases can be taken as the bases and the given bases (the interiors of the 2 cylinders P'Q' and PQ) as parts of the lateral-hypersurface.
The lateral-cylinders P'Q and PQ are congruent and lie in parallel hyperplanes, with the elements of one parallel to the elements of the other and the planes of the bases of one parallel to the corresponding planes of the bases of the other.

The visible and hidden-views in the graphic of the hypercylinder of Fig. 90 will be discussed in the next section when we make a study of the prism-cylinder.

V. PRISM-CYLINDERS AND DOUBLE-CYLINDERS

83. PLANO-CYLINDRICAL HYPERSURFACES. SECTIONS. SECTIONS. RIGHT DIRECTING-CURVES. A PLANO-CYLINDRICAL HYPERSURFACE consists of a system of parallel planes passing through the points of a plane-curve and intersecting the plane of the curve only in these points. The curve is the DIRECTING-CURVE, and the planes are the ELEMENTS. A plano-prismatic hypersurface is a particular case of a plano-cylindrical hypersurface. In this text we shall consider only the case when there is a directing-circle.
As the plano-cylindrical hypersurface is analogous to the plano-prismatic hypersurface, many of the theorems correspond. We shall state some of the corresponding theorems in a brief-way:
The INTERIOR OF THE HYPERSURFACE consists of those planes parallel to the elements which pass through the points of the interior of the directing-curve, with the usual theorems in regard to the interior. The section made by a hyperplane containing an element will be 1 or 2 elements, but the section made by a hyperplane which does not contain an element will be a cylindrical-surface, so that the hypersurface can also be regarded as a hypercylindrical-hypersurface with a directing-cylindrical-surface. Likewise, as in the case of the plano-prismatic hypersurface, a plane which does not contain a line parallel

to the elements will intersect every element in a point, and the hypersurface in a curve which can be taken as directing-curve; and the directing-curves which lie in parallel planes are congruent.

A directing-curve whose plane is absolutely-perpendicular to the planes of the elements is called a RIGHT DIRECTING-CURVE; and any plane isocline to the plane of a right directing-curve but not parallel to the elements, or what is the same thing, any plane isocline to the elements, intersects the hypersurface in a curve which is similar to the right directing-curves.

When a hypersurface has a directing-circle, the plane through its center parallel to the elements is an AXIS-PLANE OF THE HYPERSURFACE, and every point of it is a center of symmetry. When the right directing-curve is a circle, the hypersurface can be generated by the rotation of 1 of the elements around the axis-plane, that is, by the rotation of 1 of 2 parallel planes around the other. It is then a PLANO-CYLINDRICAL HYPERSURFACE OF REVOLUTION. For such a hypersurface we can say that any plane isocline to the elements, or to the axis-plane, intersects the hypersurface in a circle; and through any line which is not perpendicular to the elements nor parallel to the elements pass 2 such planes (Art. 53, Th. 1).

Theorem. Any directing-curve of a plano-cylindrical hypersurface of revolution is a directing-curve of a circular-cylindrical-surface.

84. INTERSECTION OF A PLANO-PRSIMATIC HYPERSURFACE AND A PLANO-CYLINDRICAL HYPERSURFACE. When the elements of a plano-prismatic hypersurface intersect the elements of a plano-cylindrical hypersurface only in points, the intersection of the 2 hypersurfaces consists of the lateral-surface of a set of cylinders lying in the cells of the prismatic-hypersurface and joined together by their bases, together with the curves whose interiors are these bases.

The faces of the prismatic-hypersurface are parallel planes intersecting the cylindrical-hypersurface in a set of equal parallel directing-curves of the latter, and the cells of the prismatic-hypersurface are layers, each layer intersecting the cylindrical-hypersurface in the lateral-surface of a cylinder whose bases are the interiors of 2 of these directing-curves. The interiors of the bases of the cylinders lie in the prismatic-hypersurface and in the interior of the cylindrical-hypersurface. A set of corresponding elements of the cylinders, 1 from each cylinder, are the sides of a polygon which is a directing-polygon of the prismatic-hypersurface; and the interiors of these polygons lie in the cylindrical-hypersurface and in the interior of the prismatic-hypersurface. The set of cylinders goes around the prismatic-hypersurface, while any base or interior of a cross-section of any of these cylinders is a piece cut-out of an element of the prismatic-hypersurface.

A PRISM-CYLINDER consists of the intersection of a plano-prismatic hypersurface and a plano-cylindrical hypersurface whose elements intersect only in points, together with all that portion of each hypersurface which lies in the interior of the other.

The INTERIOR OF A PRISM-CYLINDER consists of the points which are common to the interiors of its 2 hypersurfaces. A plane lying in the interior of the cylindrical-hypersurface parallel to the elements intersects the prismatic-hypersurface in a directing-polygon whose interior belongs to both hypersurfaces, and so to the interior of the prism-cylinder. The vertices of this polygon are a set of corresponding points of the bases, and the sides lie in the interiors of the set of cylinders described above.

When the elements of one hypersurface are absolutely-perpendicular to the elements of the other the prism-cylinder is a RIGHT PRISM-CYLINDER. When also the cylinders are cylinders of revolution, and when any set of corresponding elements, 1 from each cylinder, form a regular-polygon, the prism-cylinder is REGULAR.

The hypercylinder of Fig. 90 whose bases are the interiors of the 2 cylinders P'Q' and PQ is a prism-cylinder. We will write PRISM-CYLINDER P'Q'PQ to denote a hypercylinder whose bases are the interiors of cylinders, that is, whose bases are the interiors of the 2 cylinders P'Q' and PQ. We will write PARALLELOGRAM P'Q'PQ to denote a directing-parallelogram of the prismatic-hypersurface of the prism-cylinder P'Q'PQ.

The 4 cylinders PQ, P'P, P'Q, and Q'Q are joined by their bases in succession and lie in the cells of the prismatic-hypersurface. The bases of these cylinders are the interiors of circles. The intersection of the 2 hypersurfaces of the prism-cylinder P'Q'PQ consists

of the lateral-surfaces of these cylinders lying in the cells of the prismatic-hypersurface. A set of corresponding elements of the cylinders, 1 from each cylinder, are the sides of a parallelogram which is a directing-parallelogram of the prismatic-hypersurface; and the interiors of these parallelograms lie in the cylindrical-hypersurface and in the interior of the prismatic-hypersurface. In the next article we will give a more detailed discussion of the cylindrical-hypersurface of the prism-cylinder P'Q'PQ.

A plane passing through the center of the prism-cylinder P'Q'PQ parallel to the elements intersects the prism-cylinder P'Q'PQ in the parallelogram P'Q'PQ whose interior belongs to both hypersurfaces, and so to the interior of P'Q'PQ. The vertices of the parallelogram P'Q'PQ are a set of corresponding points at the center of the bases, and the sides lie in the interiors of the 4 cylinders PQ, P'P, P'Q', and Q'Q.

Refer to the description given above for further charactersitics of the prism-cylinder P'Q'PQ.

85. DIRECTING-POLYGONS AND DIRECTING-CURVES. TRIANGULAR PRISM-CYLINDERS. PRISM-CYLINDERS OF REVOLUTION. In a prism-cylinder the directing-polygons of the prismatic-hypersurface whose planes are elements of the cylindrical-hypersurface and the directing-curves of the cylindrical-hypersurface whose planes are elements of the prismatic-hypersurface are called the DIRECTING-POLYGONS and DIRECTING-CURVES OF THE PRISM-CYLINDER.

The surface of intersection of the 2 hypersurfaces is generated by moving the polygon around the curve or by moving the curve around the polygon. In the 1st case each point of the polygon moves around 1 of the cylinders; in the 2nd case each point of the curve moves along them all. The interior of the curve generates the interiors of the cylinders; the interior of the polygon generates that portion of the prism-cylinder which belongs to the cylindrical-hypersurface and to the interior of the prismatic-hypersurface. The surface of intersection is covered with the polygons and with the curves, the 2 sets forming on it a net.

The surface of intersection of the 2 hypersurfaces of the prism-cylinder of Fig. 91 is generated by moving the parallelogram AEE'A' around the circle (ABC), or by moving the circle (ABC) around the parallelogram AEE'A'. In the 1st case each point of the parallelogram AEE'A' moves around 1 of the cylinders; in the 2nd case each point of the circle moves among them all. The interior of the circle (ABC) generates the interiors of the cylinders PQ, P'P, P'Q', and Q'Q; the interior of the parallelogram AEE'A' generates that portion of the prism-cylinder P'Q'PQ which belongs to the cyclindrical-hypersurface and to the interior of the prismatic-hypersurface. The surface of intersection is covered with the parallelograms and with the circles, the 2 sets forming on it a net.

The 2 cylinders P'Q' and P'P and their interiors are visible-views in the graphic; that portion of the cylindrical-hypersurface generated by the interior of the parallelogram AEE'A' moving around the semi-circle (ABC) will be a visible-view in the graphic, and will be a $\frac{1}{2}$-lateral-hypersurface of the prism-cylinder P'Q'PQ. The interiors of the 2 cylinders PQ and Q'Q will be hidden-views in the graphic. The interiors of the 2 circles (EFG) and (A'B'C') which are 1 of the bases of the 2 cylinders PQ and Q'Q respectively, are visible-views in the graphic. The intersection of the 2 cylinders PQ and Q'Q will be a base common to both cylinders, that is,their bases are the interior of the circle (ABC), and will be a hidden-view in the graphic. The semi-circle (ABC) is a visible-view as well as that portion of the lateral-surface of the 2 cylinders PQ and Q'Q generated by the 2 sides AE and AA' of the parallelogram AEE'A' moving around the semi-circle (ABC). The other portion of the lateral-hypersurface generated by the interior of the parallelogram AEE'A' moving around the semi-circle (CDA) will be a hidden-view in the graphic, and will be the other $\frac{1}{2}$-lateral-hypersurface of the prism-cylinder P'Q'PQ. In other words, that portion of the lateral-hypersurface of the prism-cylinder P'Q'PQ generated by the interior of the parallelogram AEE'A' moving around the semi-circle (ABC) will lie on one side of the hyperplane of the parallelopiped A'E'G'C'-AEGC and will be a visible-view in the graphic, and therefore a $\frac{1}{2}$-lateral-hypersurface of the prism-cylinder P'Q'PQ; and that portion of the lateral-hypersurface generated by the interior of the parallelogram AEE'A' moving around the semi-circle (CDA) will lie on the other side of the hyperplane of the parallelopiped A'E'G'C'-AEGC and will be a hidden-view in the graphic, and therefore a $\frac{1}{2}$-lateral-hypersurface of the other portion of the prism-cylinder P'Q'PQ.

The interiors of the 2 sides AA' and EE' of the parallelogram AEE'A' will generate that portion of the lateral-hypersurface belonging to the lateral-surfaces of the 2 cylinders

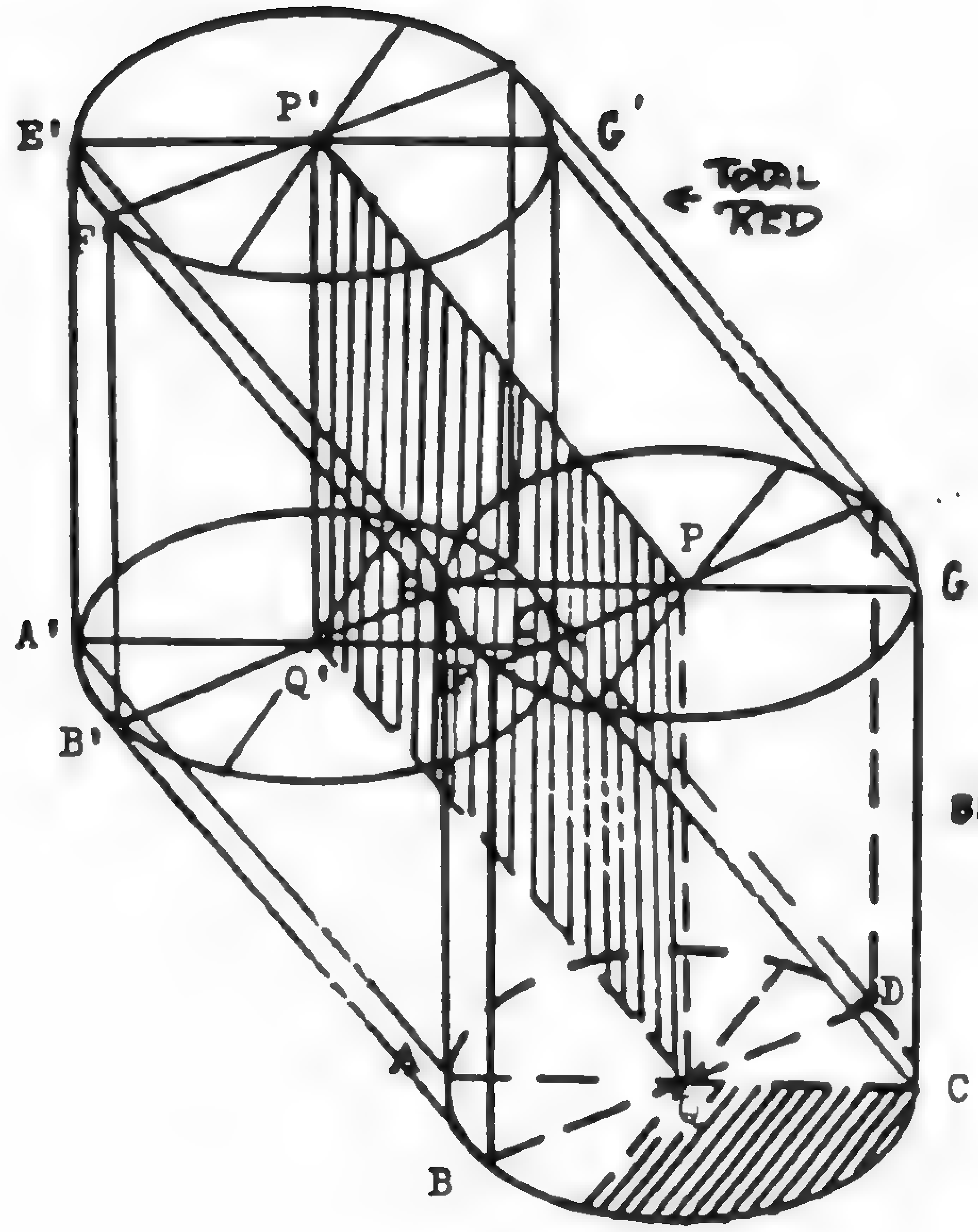


Fig. 91.

P'P and Q'Q. The visible and hidden-views for this portion of the lateral-hypersurface is the same as that described above for the 2 cylinders P'P and Q'Q.

Theorem 1. When a prism-cylinder is cut by a hyperplane passing through points of the interior and containing elements of the prismatic-hypersurface, the intersection is a cylinder, and the rest of it is separated into 2 portions, which, each combined with the cylinder and its interior, form 2 prism-cylinders whose interiors, with that of the cylinders, make-up the whole interior.

Corollary. By cutting a prism-cylinder diagonally we can form prism-cylinders in which the directing-polygons are triangles and the cylinders 3 in number, triangular prism-cylinders.

A hypercylinder whose bases are the interiors of cylinders is a prism-cylinder; and a prism-cylinder in which the directing-polygons are parallelograms (and therefore the cylinders are 4 in number) can be regarded in 2-ways as a hypercylinder. (see Fig. 91.)

When the 2 hypersurfaces of a prism-cylinder have axes-planes of symmetry, the point of intersection of these planes is a center of symmetry (as in Art. 80, Th. 2).

Theorem 2. If we rotate a right prism around the plane of one base, the rest of the prism will generate a right prism-cylinder having circles for its directing-curves. The lateral-edges generate the bases of the cylinders of the prism-cylinder, each lateral-face generates the interior of 1 of the cylinders, and the moving base generates that portion of the prism-cylinder which lies in the cylindrical-hypersurface and in the interior of the prismatic-hypersurface. The fixed-base and the interior of the prism belong to the interior of the prism-cylinder. (Fig. 91.)

In Fig. 91, let AEE'A'-QPP'Q' be a right prism. Then if we rotate this prism around the plane of the base P'Q'PQ, the rest of the prism will generate a right prism-cylinder P'Q'PQ having circles for its directing-curves. The lateral-edges AQ, EP, E'P', and A'Q' generate the bases of the cylinders of the prism-cylinder P'Q'PQ, the lateral-faces AQEP, EPE'P', E'P'A'Q', and A'Q'AQ generate the interiors of the cylinders of the prism-cylinder P'Q'PQ. The moving base AEE'A' generates that portion of the prism-cylinder

P'Q'PQ which lies in the cylindrical-hypersurface and in the interior of the prismatic-hypersurface. The fixed-base QPP'Q' and the interior of the prism AEE'A'-QPP'Q' belong to the interior of the prism-cylinder P'Q'PQ.

The plane of the base QPP'Q' of the prism AEE'A'-QPP'Q' is an axis-plane of the prism-cylinder P'Q'PQ and is aboslutely-perpendicular to the plane of the directing-circle (ABC). at the point Q, the center of the circle (ABC).

86. INTERSECTION OF 2 PLANO-CYLINDRICAL HYPERSURFACES. When the elements of a plano-cylindrical hypersurface intersect the elements of a 2nd plano-cylindrical hypersurface only in points, each element of one hypersurface intersects the other hypersurface in a directing-curve, and the surface of intersection consists of the curves of either of these sets. The interiors of the curves of each set lie in one of the hypersurfaces and in the interior of the other.

A DOUBLE-CYLINDER consists of the intersection of 2 plano-cylindrical hypersurfaces whose elements intersect only in points, together with that portion of each which lies in the interior of the other. The directing-curves of each hypersurface whose planes are elements of the other are called the DIRECTING-CURVES OF THE DOUBLE-PRISM.

The INTERIOR OF THE DOUBLE-PRISM consists of the points which are common to the interiors of its 2 hypersurfaces. A plane lying in the interior of one hypersurface parallel to its elements intersects the other hypersurface in a directing-curve whose interior belongs to the interior of the double-cylinder.

When the elements of one hypersurface of a double-cylinder are absolutely-perpendicular to the elements of the other the double-cylinder is a RIGHT DOUBLE-CYLINDER.

The surface of intersection of the 2 hypersurfaces is generated by moving the directing-curve of one system around a directing-curve of the other.

87. THEOREMS ON CYLINDERS OF DOUBLE-REVOLUTION.

Theorem 1. If we rotate a cylinder of revolution around the plane of one base, the rest of the cylinder will generate a right double-cylinder with directing-circles; and the double-cylinder can be generated in 2-ways by the rotation of a cylinder of revolution around one of its bases.

The right double-cylinder with directing-circles is therefore called a DOUBLE-CYLINDER OF DOUBLE-REVOLUTION.

Theorem 2. In a cylinder of double-revolution the intersection of the 2 hypersurfaces lies in a hypersphere, and in this hypersphere is a surface of double-revolution.

Theorem 3. Conversely, any surface of double-revolution in a hypersphere is the surface of intersection of the 2 hypersurfaces of a cylinder of double-revolution.

The cylinders of double-revolution can be regarded as inscribed in the hypersphere.

[illegible] and hypervolume of a hypersphere are then given in terms of an analytic- [illegible] and a [illegible]—this part of the [illegible] will [illegible] all present-day mathematicians and scientists—[illegible] Einstein [illegible].
[illegible] was severely-limited by resorting to analytic-solutions of physics-problems [illegible] uses [illegible] as well as ANALYTIC-SOLUTIONS in [illegible] powerful-math tool which I call [illegible] made-up to form a [illegible] having these 3 [illegible]-components: [illegible]-component (axiomatic), [illegible]-component, and a [illegible]-component. [illegible] discovered and develope[d] [illegible]-approach to geometry, then came Descartes with his discovery of Analyti[c] [illegible] algebra. FROM #20 LETTERS REP.

CHAPTER VIII

MEASUREMENT OF VOLUME AND HYPERVOLUME IN HYPERSPACE

I. VOLUME

88. LATERAL-VOLUMES OF HYPERPRISMS AND HYPERPYRAMIDS. VOLUME OF THE DOUBLE-PRISM. The cells of the polyhedroids that we have studied are polyhedrons of solid-geometry, and it is only necessary to state the theorems which concern their volumes.

Theorem 1. The lateral-volume of a hyperprism is equal to the area of the surface of a right-section multiplied by the lateral-edge.

Theorem 2. The lateral-volume of a right hyperprism is equal to the area of the surface of the base multiplied by the altitude.

It should be understood that the area of a polyhedron is the area of the surface of the polyhedron whose interior is the base.

Theorem 3. The lateral-volume of a regular-hyperpyramid is equal to the area of the surface of the base multiplied by 1/3 of the slant-height, the common-altitude of the lateral-pyramids.

Theorem 4. The lateral-volume of a frustrum of a regular-hyperpyramid is equal to the sum of the surface-areas of the bases plus a mean-proportional between them, multiplied by 1/3 of the slant-height.

Theorem 5. In a double-prism the total-volume of one set of prisms is equal to the common-area of their bases multiplied by the perimeter of a right directing-polygon of the hypersurface around which the set of prisms extends.

Proof: Any prism of the given set has its bases in the 2 faces of a cell of the hypersurface around which this set of prisms extends, and its altitude is the distance between these 2 faces. Now a right directing-polygon of the hypersurface is a polygon whose plane is absolutely-perpendicular to the elements, and the sides which lie in this cell is perpendicular to the faces and measures the distance between them. Therefore the volume of this prism is equal to the area of its base multiplied by this side of the right directing-polygon of the hypersurface, and the total-volume of the given set of prisms is equal to the common-area of the bases multiplied by the perimeter of the right directing-polygon.

Corollary. The total-volume of a right double-prism is equal to the area of one directing-polygon multiplied by the perimeter of the other, plus the area of the 2nd multiplied by the perimeter of the 1st.

89. LATERAL-VOLUMES OF CYLINDRICAL AND CONICAL-HYPERSURFACES. In the case of curved-hypersurfaces we have to employ the theory of limits or some other equivalent theory and extend our definition of volume. In section III of this chapter we shall make use of the calculus in deriving the volume of a hypersphere.

Theorem 1. The lateral-volume of a right spherical-hypercylinder is equal to the area of the base multiplied by the altitude. Its formula is

$$V = 4\pi r^2 h,$$

r being the radius and h the altitude.

In Fig. 88, $OA = r$, and for the altitude of the spherical-hypercylinder S'S, we can take $O'O = h$, with the understanding that O'O is perpendicular to the hyperplane of the sphere S at the point O, in which case the above formula applies.

Theorem 2. The lateral-volume of a right spherical-hypercone is equal to the area of the base multiplied by 1/3 of the slant-height. Its formula is

$$V = \frac{4\pi}{3} r^2 h',$$

h' being the slant-height.

Theorem 3. The lateral-volume of a frustrum of a right spherical-hypercone is given by the formula

$$V = h'(r^2 + rr' + r'^2),$$

r' being the radius of the upper-base.

Theorem 5. The total-volume of a cylinder of double-revolution is equal to the area of one directing-circle multiplied by the circumference of the other, plus the area of the 2nd multiplied by the circumference of the 1st. Its formula is

$$V = 2\pi^2 rr'(r + r'),$$

r and r' being the radii of the 2 circles.

II HYPERVOLUME

89. USAGE OF THE TERMS HYPERSOLID AND HYPERVOLUME. We shall use the term HYPERSOLID for that portion of hyperspace which constitutes the interior of the polyhedroid or a simple closed-hypersurface such as a hypercone, hypersphere, or double-cylinder. A hypersolid has HYPERVOLUME which can be computed from the measurements of certain segments and angles, and which can be expressed in terms of the hypervolume of a given hypercube taken as unit. We shall use freely the forms of expression commonly employed in measuration. The distinction between hypersurface and hypersolid is important, but we shall use these terms interchangeably, speaking, for example, of the hypervolume of a given hypersurface, and, on the other hand, of the vertices, edges, faces, or cells of a hypersolid.

By the RATIO OF 2 HYPERSOLIDS we mean the ratio of their hypervolumes. Thus the ratio of any hypersolid to the unit-hypercube is the same as the hypervolume of the hypersolid. 2 hypersolids which have the same hypervolume are EQUIVALENT; and if a hypersolid is divided into 2 or more parts, the hypervolume of the whole is equal to the sum of the hypervolumes of the parts. 2 hypersolids which are congruent are equivalent.

90. CONGRUENT AND EQUIVALENT HYPERPRISMS.

Theorem 1. 2 right hyperprisms are congruent when they have congruent bases and equal altitudes.

For a given base of one can be made to coincide with either base of the other, and in 1 of these 2 positions the hyperprisms will lie on the same side of the hyperplane of the coinciding bases and will coincide throughout.

Theorem 2. An oblique-hyperprism is equivalent to a right hyperprism having for its base a right-section and for its altitude a lateral-edge of the oblique-hyperprism.

For we can take a right-section of the oblique-hyperprism and construct a right-hyperprism on the oblique-hyperprism by extending the elements of the oblique-hyperprism from a right-section, with the altitude of the right-hyperprism equal to a lateral-edge of the oblique-hyperprism, and the right-section as 1 of its bases. We then prove by superposition that the 2 hyperprisms are equivalent.

Corollary. Any 2 hyperprisms cut from the same prismoidal-hypersurface with equal lateral-edges are equivalent.

91. HYPERVOLUME OF A HYPERPARALLELOPIPED.

Theorem 1. The hypervolume of a rectangular-hyperparallelopiped is equal to the product of its 4 dimensions.

For 2 rectangular-hyperparallelopipeds having congruent-bases are to each other as their altitudes, since this can be proved when the altitudes are commensurable and then when the altitudes are incommensurable. Then we prove that when they have 2 dimensions in common they are to each other as the products of the other 2 dimensions; when they have 1 dimension in common they are to each other as the products of the other 3 dimensions; and finally, in any case, they are to each other as the products of their 4 dimensions. From the last statement by taking for the 2nd hyperparallelopiped the unit-hypercube, we have the theorem as stated.

Theorem 4. The hypervolume of any hyperparallelopiped is equal to the volume of any base multiplied by the corresponding altitude.

The proof of the theorem is the analogue to the corresponding proof of the theorem for the volume of a parallelopiped in the solid-geometry. We prove the theorem by constructing an equivalent hyperparallelopiped with base equivalent to the base and altitude equal to the altitude of the given hyperparallelopiped. In order to construct an equivalent hyperparallelopiped which is rectangular we need to construct 3 hyperparallelopipeds in succession, using the given hyperparallelopiped to construct the 2nd, then the 2nd to construct the 3rd, and finally the 3rd to construct the 4th. In all these different constructions we extend the edges of 1 hyperparallelopiped to construct the next in succession. The bases of all 4 of the hyperparallelopipeds will be equivalent, and the altitude of the 4th hyperparallelopiped, being rectangular, will be equal to the altitude of the 1st.

92. HYPERVOLUME OF ANY HYPERPRISM.

Theorem 1. The hypervolume of a hyperprism whose base is the interior of a prism is equal to the volume of its base multiplied by its altitude. (Fig. 92.)

As any prism can be divided into a triangular-prism, it is only necessary to prove the theorem when the base is a triangular-prism.

In the proof that follows, we shall use a special-notation for hyperprisms whose bases are the interiors of triangular-prisms or parallelopipeds. We shall write HYPERPRISM AEE'A'-ABD to denote a hyperprism whose base is the interior of a triangular-prism EFH-ABD, or what is the same thing, a hyperprism whose directing-polygons of its 2 hypersurfaces are a parallelogram AEE'A' and a triangle ABD, where the triangle ABD moving around the parallelogram AEE'A' will have its interior generate the bases of the hyperprism AEE'A'-ABD. We shall write PARALLELOPIPED EFGH-ABCD to denote a parallelopiped whose bases are the interiors of parallelograms. We shall write HYPERPARALLELOPIPED AEE'A'-ABCD to denote a hyperparallelopiped whose base is the interior of a parallelopiped EFGH-ABCD.

Given: A hyperprism AEE'A'-ABD with altitude h and base EFH-ABD.

To Prove: The hypervolume of the hyperprism AEE'A'-ABD is equal to the volume of its base EFH-ABD multiplied by its altitude h.

Proof: On the triangular-prism EFH-ABD we build a parallelopiped EFGH-ABCD by joining an equal triangle DBC to its base so as to form a parallelogram ABCD, and drawing a 4th lateral-edge CG. We join to the triangular-prism EFH-ABD a 2nd triangular-prism HFG-DBC and the 2 are symmetrically-situated with respect to the center of the parallelopiped EFGH-ABCD, and therefore are equal (Art. 48, Th.). On the hyperprism AEE'A'-ABD we then build a hyperparallelopiped AEE'A'-ABCD having EFGH-ABCD as base. We join to the hyperprism AEE'A'-ABD a 2nd hyperprism CGG'C'-DBC with triangular-prism HFG-DBC as base, and the 2 hyperprisms AEE'A'-ABD and CGG'C'-DBC are congruent, since the center of the hyperparallelopiped AEE'A'-ABCD lies in the diagonal-hyperplane of the diagonal-parallelopiped H'F'D'B'-HFDB along which the 2 hyperprisms AEE'A'-ABD and CG'C'-DBC are joined (Art. 74, Th.1, and Art. 49, Th. 2). The hypervolume of the hyperprism AEE'A'-AB~~C~~D is therefore equal to $\frac{1}{2}$ of the hypervolume of the hyperparallelopiped AEE'A'-ABCD, and so the volume of its own base EFH-ABD multiplied by its altitude h. In other words, the hypervolume of the hyperprism AEE'A'-ABD with base EFH-ABD is

$$V_4 = (\text{volume of EFH-ABD})\, h = (\text{base})\ (\text{altitude}) = Bh. \quad (\text{Q.E.D.})$$

In another way, a diagonal-hyperplane passing through 2 opposite-faces of a hyperparallelopiped divides the hyperparallelopiped into 2 equivalent hyperprisms whose bases are the interiors of triangular-prisms. The base of the hyperparallelopiped is divided into 2 triangular-prisms of equal volume, and the altitudes of the 2 triangular-prisms are equal. It follows then that the hypervolume of a hyperprism whose base is the interior of a prism is equal to the volume of its base multiplied by its altitude. Since any hyperprism can be divided into hyperprisms whose bases are the interiors of

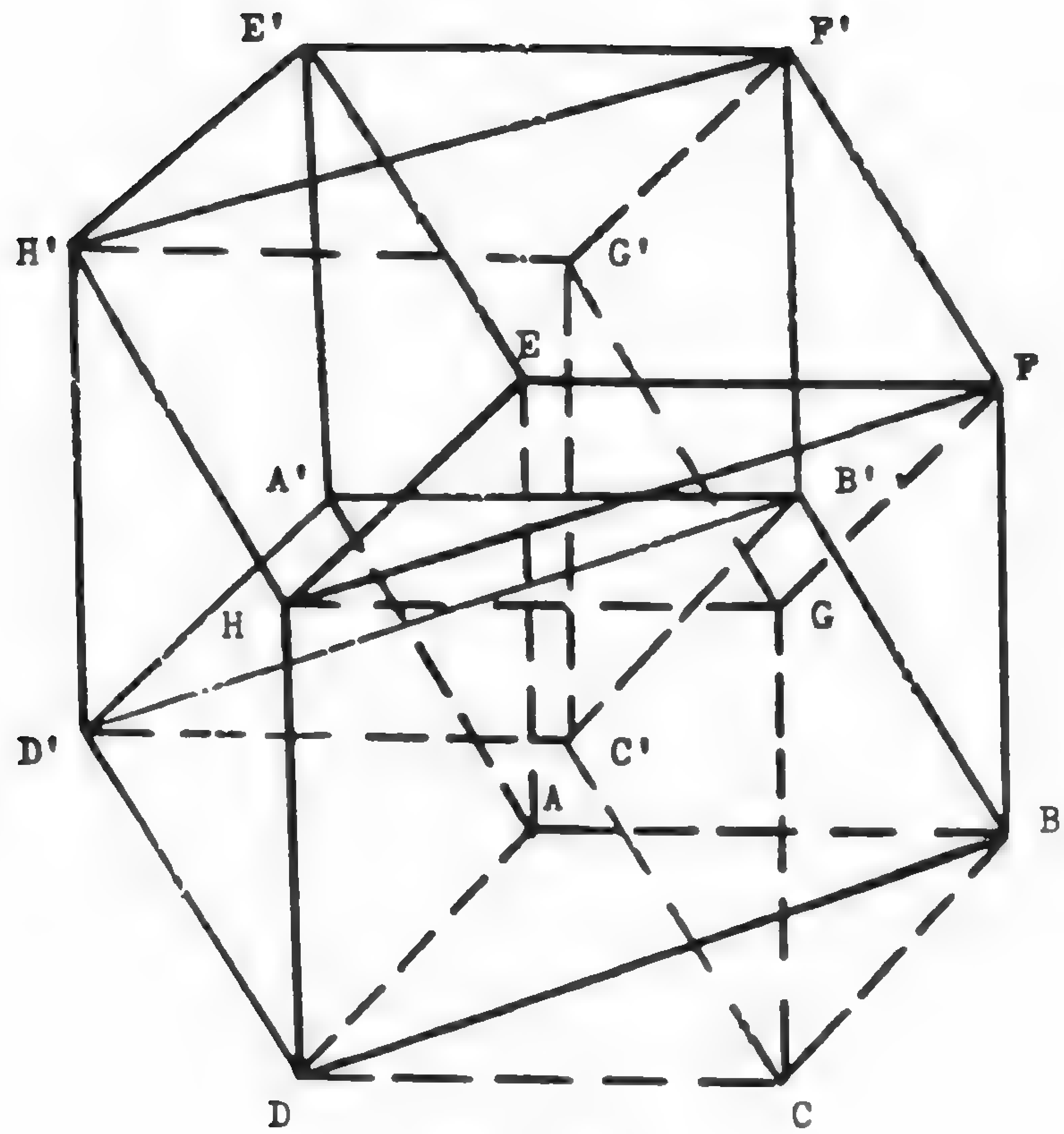


Fig. 92.

triangular-prisms having as altitudes the altitude of the hyperprism, and therefore the hypervolume of any hyperprism is equal to the product of its base and altitude.

Theorem 2. 2 right hyperprisms are equivalent if they have equal altitudes and if their bases are the interiors of tetrahedrons which can be so placed that as triangular-pyramids they shall have equivalent bases and equal altitudes.

Theorem 3. The hypervolume of any hyperprism is equal to the volume of its base multiplied by its altitude.

Corollary. The hypervolume of a hyperprism is equal to the volume of a right-section multiplied by the lateral-edge.

93. HYPERVOLUME OF A HYPERPYRAMID.

Theorem 1. A hyperplane-section of a hyperpyramid parallel to the base is similar to the base, and its volume is proportional to the cube of its distance from the vertex.

Theorem 2. If a hyperpyramid is cut by a hyperplane parallel to its base, the lateral-edges and the altitude are divided proportionally, and the section formed is similar to the base.

Theorem 3. Parallel-sections of a hyperpyramid are to each other as the cubes of their distances from the vertex. (Fig. 93.)

By dividing the parallel-sections into tetrahedrons we can divide the hyperpyramid into parallel-sections of pentahedroids, all of which have the same altitude and the ratios of their volumes equal to a proportionality-constant. Therefore, it is only necessary to prove the theorem when the parallel-sections cells of tetrahedrons, that is, for parallel-sections of a pentahedroid.

Given: A hyperpyramid V-ABCD with parallel-sections EFGH and ABCD whose distances from the vertex are VH and VD, respectively.

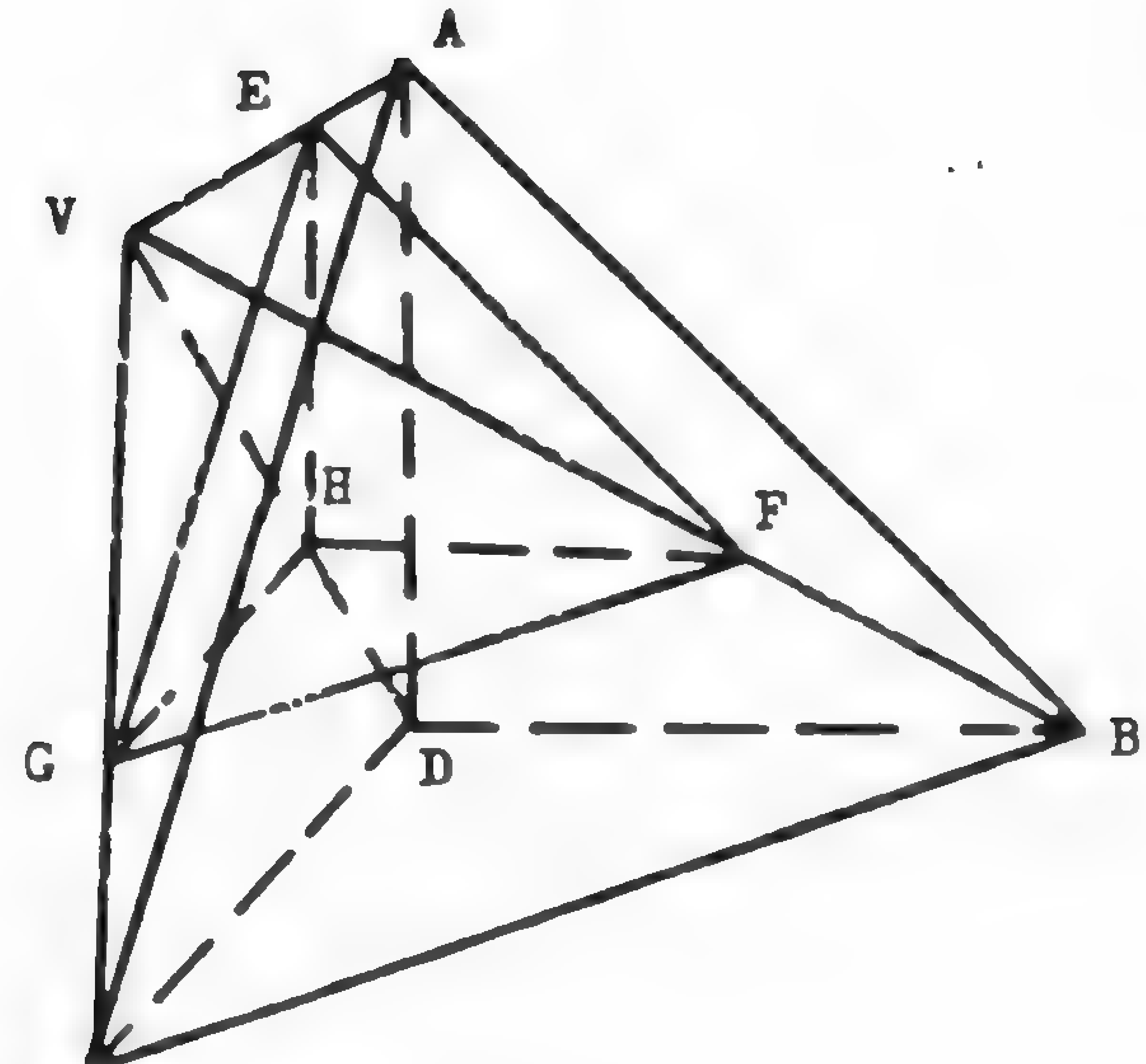


Fig. 93.

To Prove: $\dfrac{\text{Volume EFGH}}{\text{Volume ABCD}} = \dfrac{\overline{VH}^3}{\overline{VD}^3}$.

Proof: The 2 tetrahedrons EFGH and ABCD are similar, since parallel-sections of the hyperpyramid V-ABCD parallel to the base ABCD cut-out similar figures (Th. 2). Now by a theorem of the solid-geometry, the volumes of similar-tetrahedrons are in the same ratio as the cubes of their corresponding edges, and we have

$$\frac{\text{Volume EFGH}}{\text{Volume ABCD}} = \frac{\overline{FH}^3}{\overline{BD}^3}.$$

By similar-triangles, $\dfrac{VH}{VD} = \dfrac{VF}{VB} = \dfrac{FH}{BD}$. The substitution of $\dfrac{FH}{BD}$ in the above expression gives us the desired result, and therefore

$$\frac{\text{Volume EFGH}}{\text{Volume ABCD}} = \frac{\overline{VH}^3}{\overline{VD}^3}. \quad \text{(Q.E.D.)}$$

Theorem 4. If 2 hyperpyramids have equivalent bases and equal altitudes, hyperplane-sections parallel to the bases and at the same distance from the vertices are equivalent.

Theorem 5. 2 pentahedroids are equivalent if they can be so placed that as hyperpyramids they shall have equivalent bases and equal altitudes.

By dividing the altitude into n equal parts, we can construct a series of inscribed and circumscribed hyperprisms and prove that the hypervolume of either pentahedroid is the limit of the sum of the hypervolumes of the set of hyperprisms inscribed or circumscribed to it when the number n of subdivisions of the altitude is increased indefinitely. Thus we prove the theorem in the same manner as we prove the corresponding theorem in the solid-geometry.

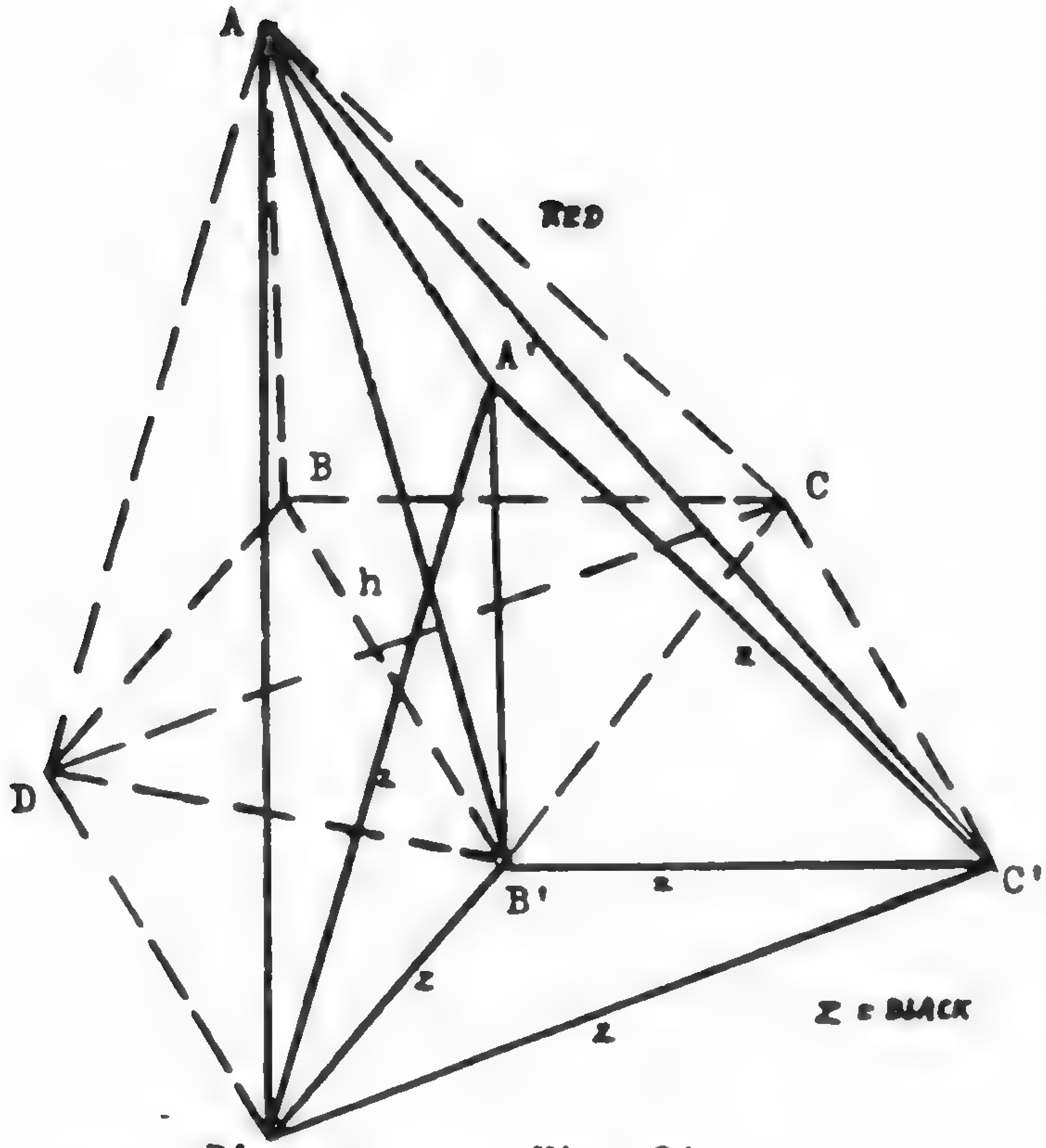
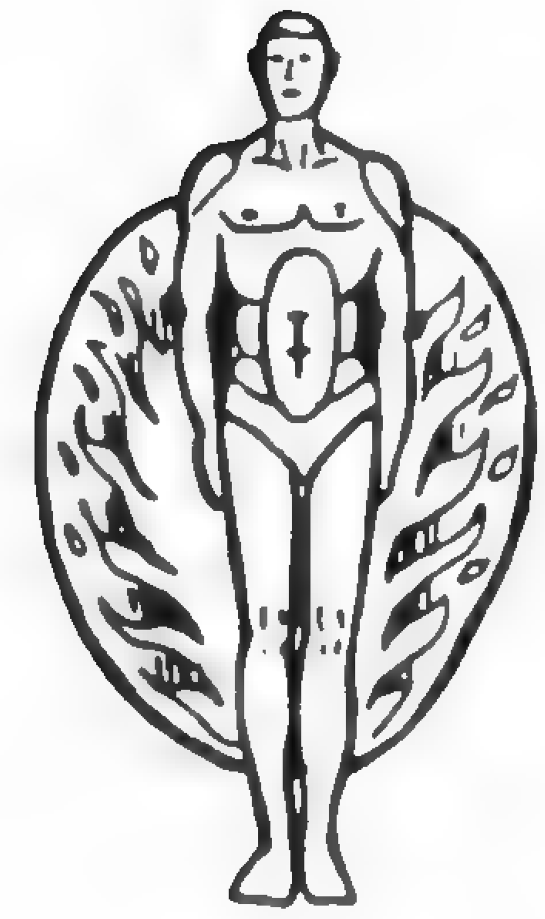


Fig. 94.



Theorem 6. The hypervolume of any hyperpyramid is equal to the volume of its base multiplied by $\frac{1}{4}$ of its altitude. (Fig. 94.)

As any polyhedron can be divided into tetrahedrons, it is only necessary to prove the theorem when the base is the interior of a tetrahedron, that is, to prove it for pentahedroids.

Given: A pentahedroid AA'B'C'D' whose base is the interior of a tetrahedron A'B'C'D' and whose altitude is h.

To prove: The hypervolume of AA'B'C'D' is equal to the volume of its base A'B'C'D' multiplied by $\frac{1}{4}$ of its altitude h.

Proof: Drawing lines through B', C', and D' parallel to AA', and a hyperplane through A parallel to the hyperplane of the tetrahedron A'B'C'D', we have a hyperprism ABCD-A'B'C'D' composed of the pentahedroid AA'B'C'D' and the hyperpyramid A-BCDB'C'D'. The hyperpyramid A-BCDB'C'D' is divided into 3 equivalent pentahedroids ABCDB', AB'CDD', and AB'C'D'C by dividing the triangular-prism BCD-B'C'D' into 3 equivalent tetrahedrons BCDB', B'CDD', and B'C'D'C. The common altitude of the 3 pentahedroids being the distance of the vertex A from the hyperplane of the triangular-prism BCD-B'C'D', that is, AA' = h. Now of these 3 pentahedroids, the pentahedroid ABCDB' can be regarded as having ABCD as its base, and as its vertex the point B' of the lower-base A'B'C'D' of the hyperprism ABCD-A'B'C'D'. Regarded in this way, the pentahedroid ABCDB' is seen to be equivalent to the original pentahedroid AA'B'C'D', since the bases of the 2 pentahedroids AA'B'C'D' and ABCDB' are the bases of the hyperprism ABCD-A'B'C'D', and their common altitude h is the altitude of the hyperprism ABCD-A'B'C'D'. The pentahedroid AA'B'C'D' is, therefore, 1 of 4 equivalent pentahedroids which go to make up the hyperprism ABCD-A'B'C'D'; and its hypervolume is $\frac{1}{4}$ of the hypervolume of the hyperprism ABCD-A'B'C'D', and so equal to the volume of its own base A'B'C'D' multiplied by $\frac{1}{4}$ of its altitude h. Accordingly, the hypervolume of the pentahedroid AA'B'C'D' is

$$V_4 = \frac{h}{4} \text{ (volume of A'B'C'D')} = \tfrac{1}{4} \text{ (base) (altitude)} = \tfrac{1}{4}Bh,$$

B being the base, and h its altitude. (Q.E.D.)

94. HYPERVOLUME OF A FRUSTRUM OF A HYPERPYRAMID.

Theorem. The hypervolume of a frustrum of a hyperpyramid is given by the formula

$$V_4 = \tfrac{1}{4}h(B + B^{2/3}b^{1/3} + B^{1/3}b^{2/3} + b),$$

where B and b are the volumes of the bases and h is the altitude. (Fig. 95.)

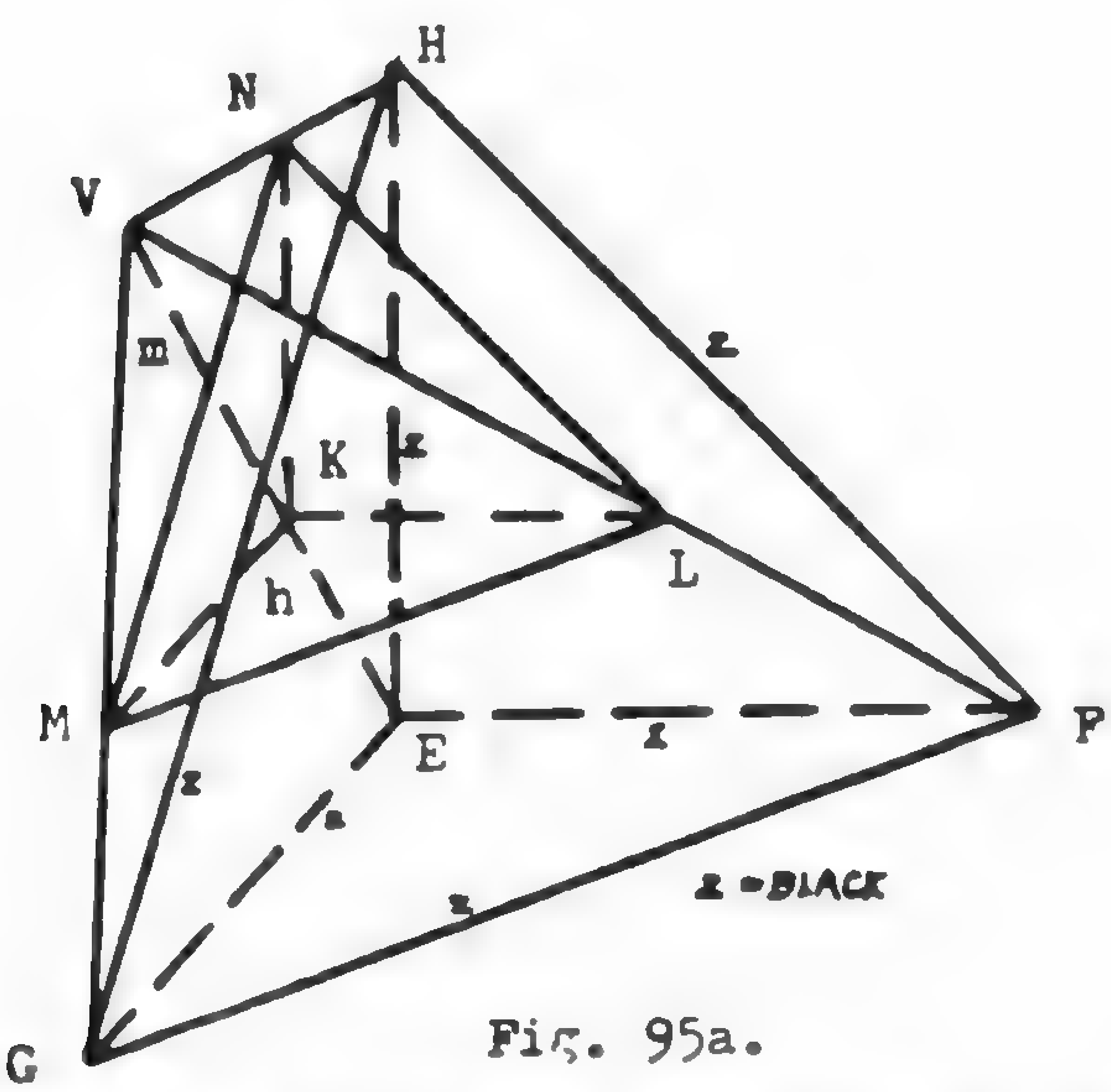


Fig. 95a.

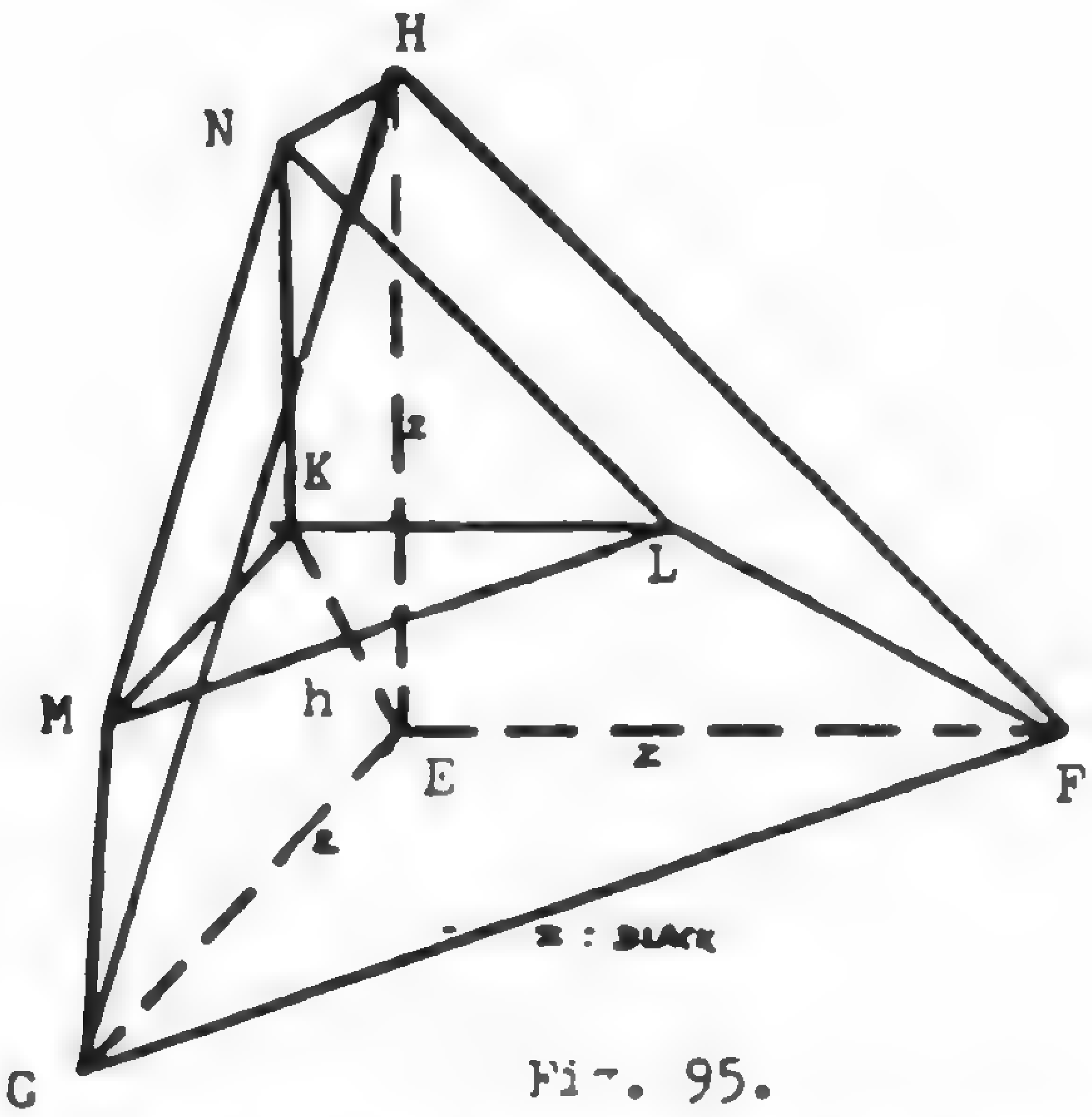


Fig. 95.

By dividing the bases into tetrahedrons we can divide the frustrum into frustrums of pentahedroids, in which h has the same value and the ratio of their volumes equal to a proportionality-constant which has the same value for each.

Given: The frustrum KLMN-EFGH of the hyperpyramid (pentahedroid) V-EFGH with upper-base KLMN denoted by b, lower-base EFGH denoted by B, and altitude h.

To Prove: $$V_4 = \tfrac{1}{4}h(B + B^{2/3}b^{1/3} + B^{1/3}b^{2/3} + b).$$

Proof: Complete the hyperpyramid V-EFGH (Fig. 95a.), of which KLMN-EFGH is a frustrum; represent the altitude of the hyperpyramid V-KLMN by m, where m = VK; then the altitude of V-EFGH is h + m.

The hypervolume of the frustrum equals the difference between the hypervolumes of the hyperpyramids V-EFGH and V-KLMN. Therefore,

$V_4 = \tfrac{1}{4}B(h + m) - \tfrac{1}{4}bm$. But $\dfrac{b}{m} = \dfrac{m^3}{(h + m)^3}$, or $\dfrac{b^{1/3}}{B^{1/3}} = \dfrac{m}{h + m}$; whence $m = \dfrac{h\,b^{1/3}}{B^{1/3} - b^{1/3}}$;

The substitution of this value of m in the formula for V_4 above gives

$$V_4 = \tfrac{1}{4}Bh(1 + b^{1/3}/B^{1/3} - b^{1/3}) - \tfrac{1}{4}bh\,\frac{b^{1/3}}{B^{1/3} - b^{1/3}} = \tfrac{1}{4}h(B^{4/3} - b^{4/3}/B^{1/3} - b^{1/3}).$$

The binomial in the numerator can be written as the difference of 2 hypercubes in the form $(B^{1/3})^4 - (b^{1/3})^4$, which factors into $(B^{1/3} - b^{1/3})(B + B^{2/3}b^{1/3} + B^{1/3}b^{2/3} + b)$.

Division of the numerator and the denominator by the common binomial-factor $B^{1/3} - b^{1/3}$ and simplification of the polynomial-factor leads to the required result; namely,

$$V_4 = \tfrac{1}{4}h(B + B^{2/3}b^{1/3} + B^{1/3}b^{2/3} + b). \quad \text{(Q.E.D.)}$$

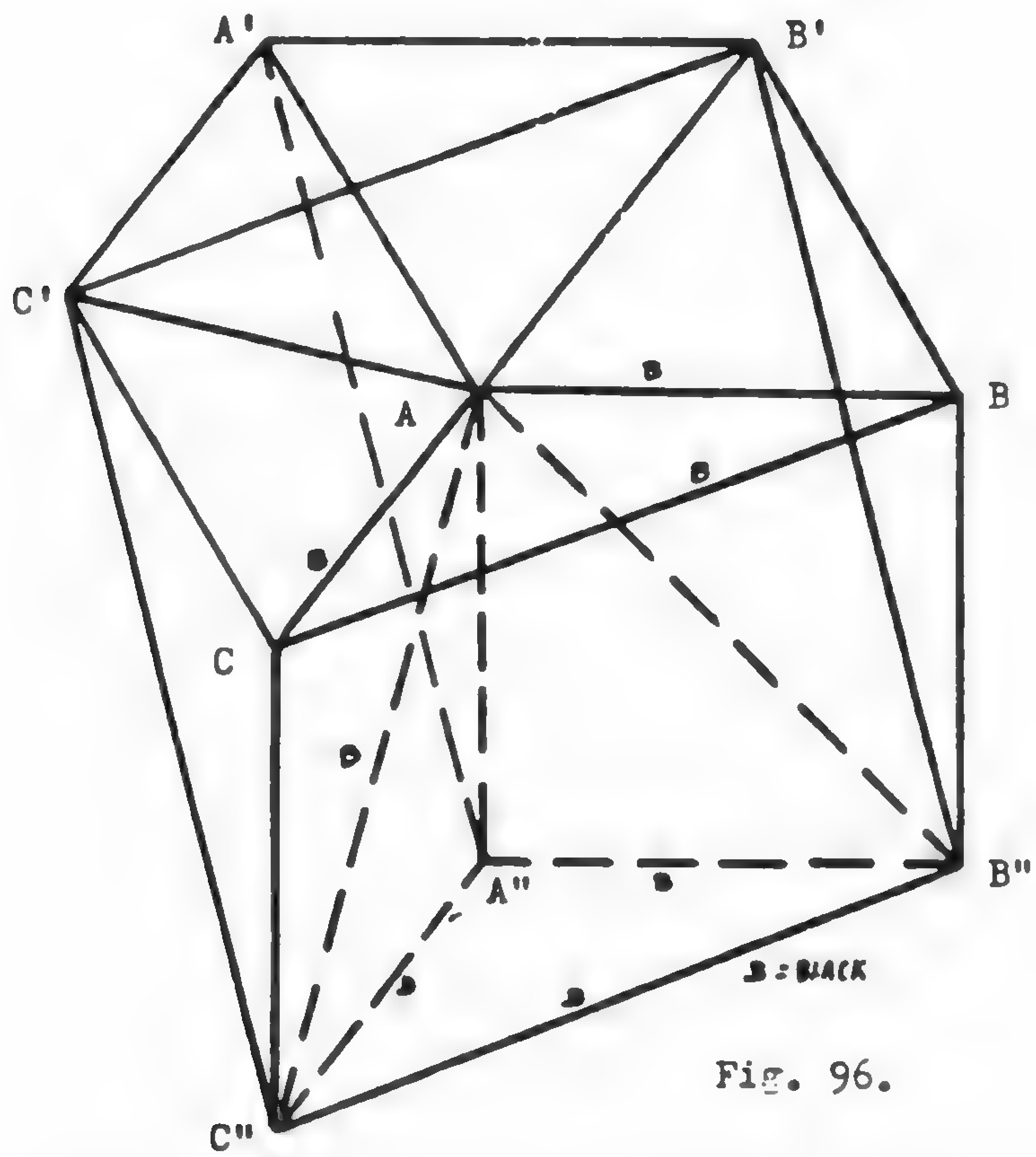
In the proof of the above theorem we have expressed the cube-root of a number as a fractional-exponent instead of a radical-sign.

95. HYPERVOLUME OF A DOUBLE-PRISM.

Theorem 1. The hypervolume of a doubly-triangular prism is equal to 6 times the hypervolume of the pentahedroid whose vertices are the points obtained by taking the vertices of a base in one of the 2 sets of prisms together with the vertices of a base in the other set. (Fig. 96.)

Given: A doubly-triangular prism AA'A"-A"B"C".

To Prove: The hypervolume of the double-prism AA'A"-A"B"C" is equal to 6 times the hypervolume of the pentahedroid whose vertices are the points obtained by taking the vertices of a base in one of the 2 sets of prisms together with the vertices of a base in the other set.

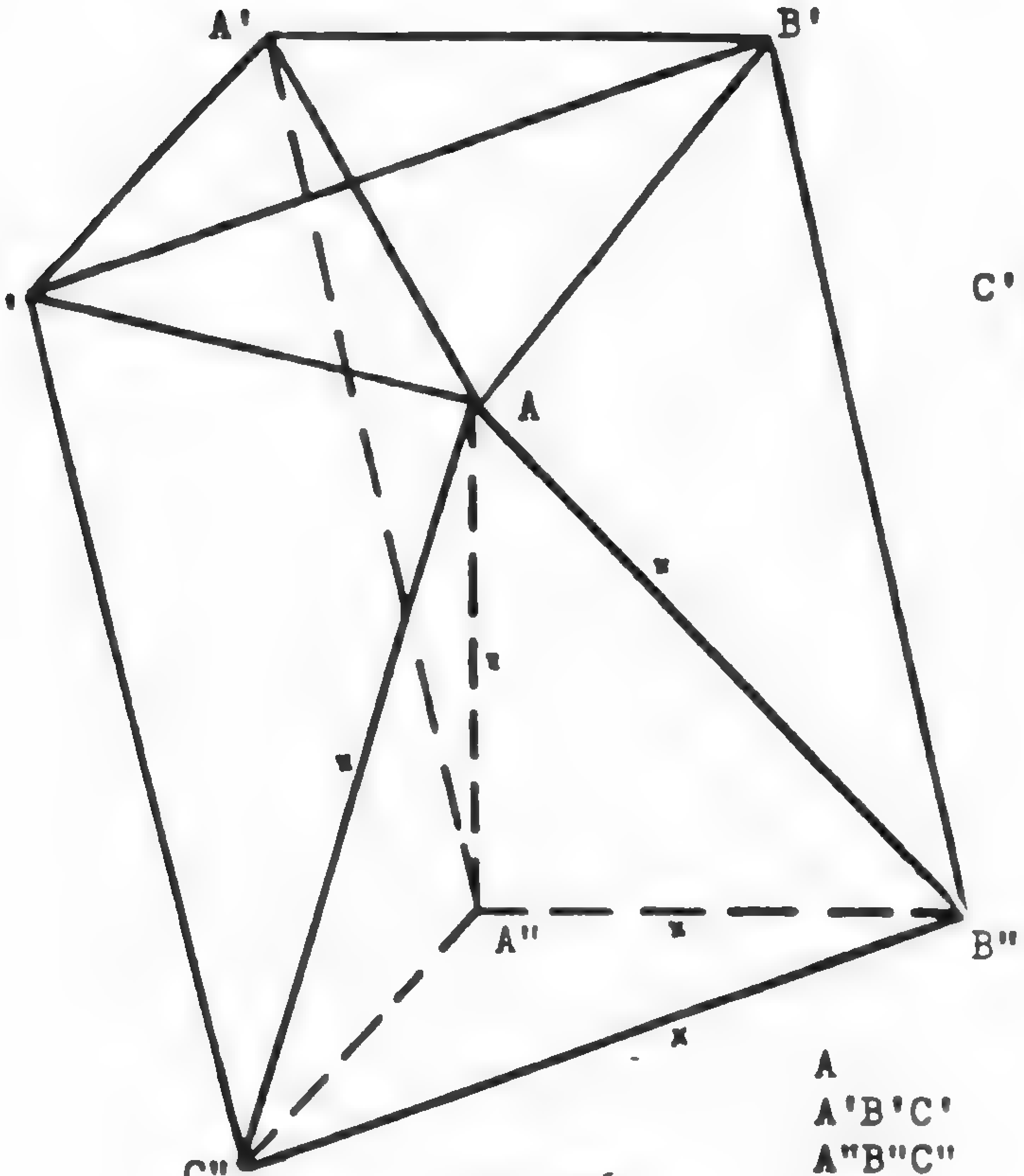


A B C
A'B'C'
A"B"C"
(Matrix-Form)

Fig. 96.

Proof: The point A and the plane of the parallelogram B'C'B"C" determine a hyperplane which divides the double-prism AA'A"-A"B"C" into 2 hypersolids A'A"-AB'C'B"C" (Fig. 96a) and BC-AB'C'B"C" (Fig. 96b). The hyperplane cannot contain any of the points B, C, A', or A"; for it contains at least 1 vertex of each of the 6 triangles of the double-prism AA'A"-A"B"C", and if it contained any 1 of these triangles, it would contain the 2 triangles parallel to it and so all the 9 vertices of the double-prism AA'A"-A"B"C". Now the hyperplane intersects the hyperplane of the triangular-prism ABC-A'B'C' in the plane of the triangle AB'C' which separates B and C from A', and it intersects the hyperplane of the triangular-prism ABC-A"B"C" in the plane of the triangle AB"C" which separates B and C from A". Therefore B and C are separated in hyperspace from A' and A" by this hyperplane (Art. 5).

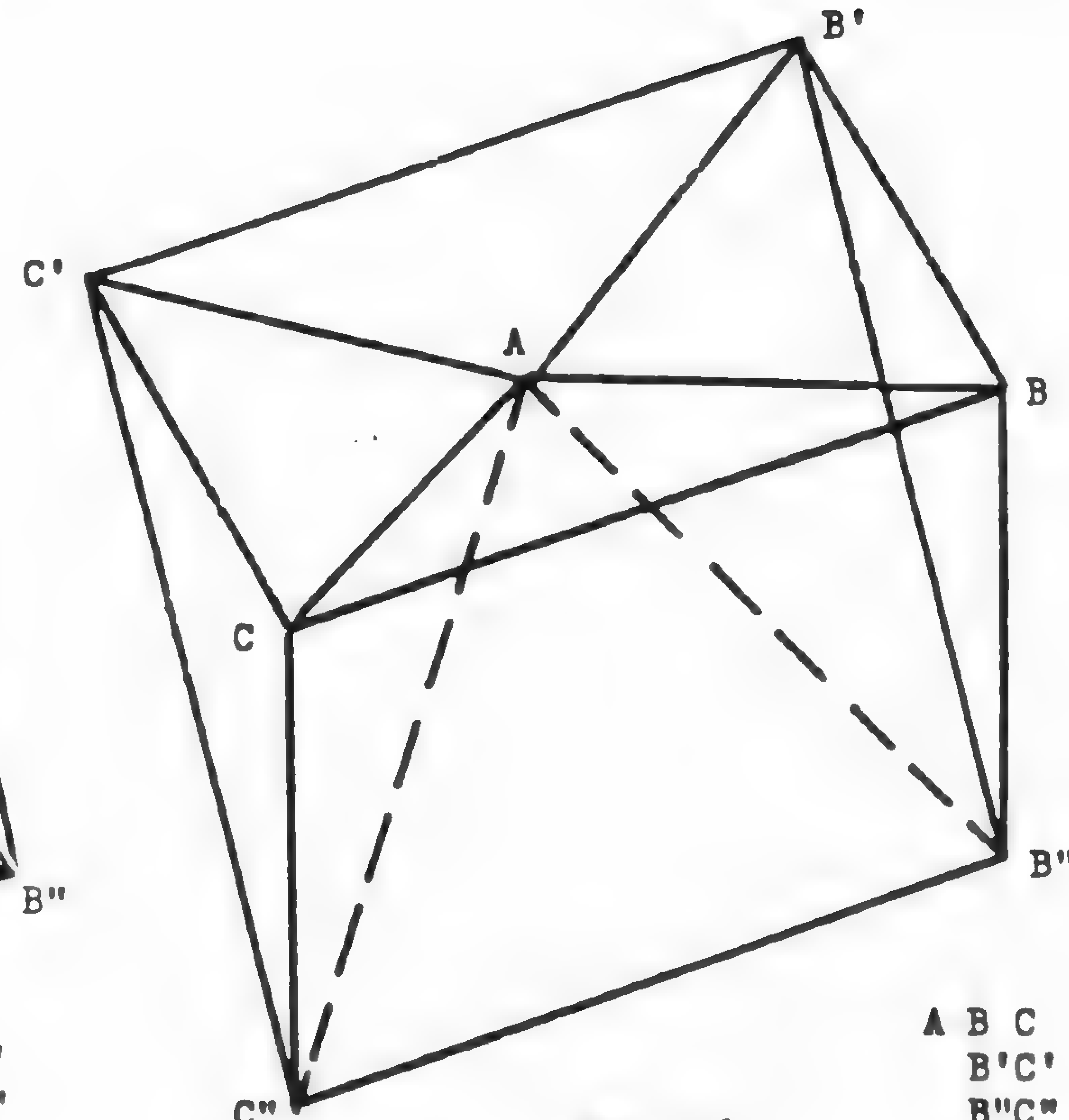
The 2 hypersolids A'A"-AB'C'B"C" and BC-AB'C'B"C" are hyperpyramids each having its vertex at A and a triangular-prism for base, and we can denote these 2 hyperpyramids by A-A'B'C'A"B"C" (Fig. 96a) and A-BB'B"CC'C" (Fig. 96b), each having vertex A, and triangular-prisms A'B'C'-A"B"C" and BB'B"-CC'C", respectively. Each of the hyperpyramids A-A'B'C'A"B"C" and A-BB'B"CC'C" can be divided into 3 equivalent pentahedroids, the base being divided into 3 equivalent tetrahedrons. This can be done in such a way that 1 of the pentahedroids of the 1 st set shall be the pentahedroid ABCC'C" (Fig. 96bb') and 1 of the pentahedroids of the 2nd set the pentahedroid AA'A"B"C" (Fig. 96aa').



A
A'B'C'
A"B"C"

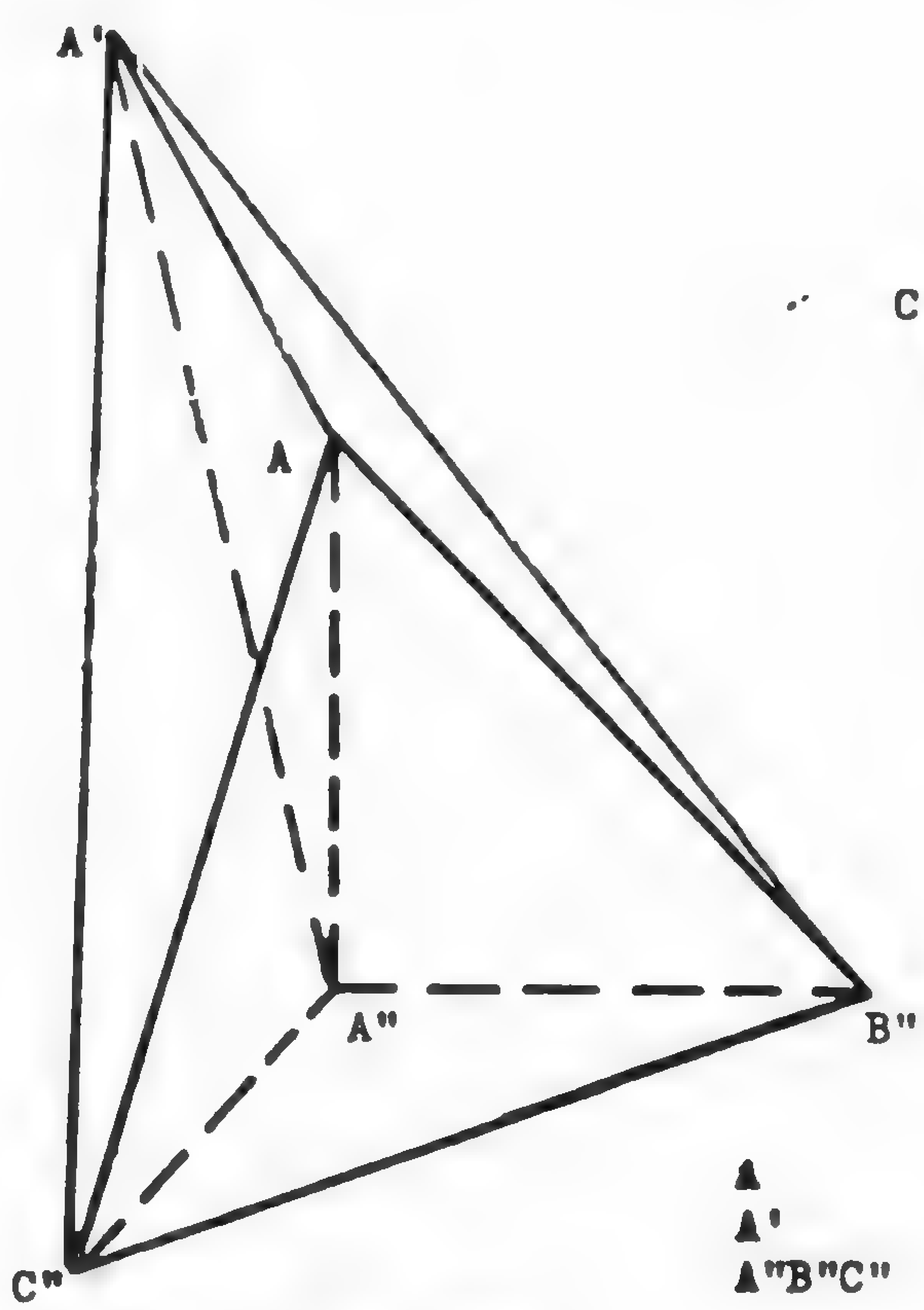
Fig. 96a

x = BLACK



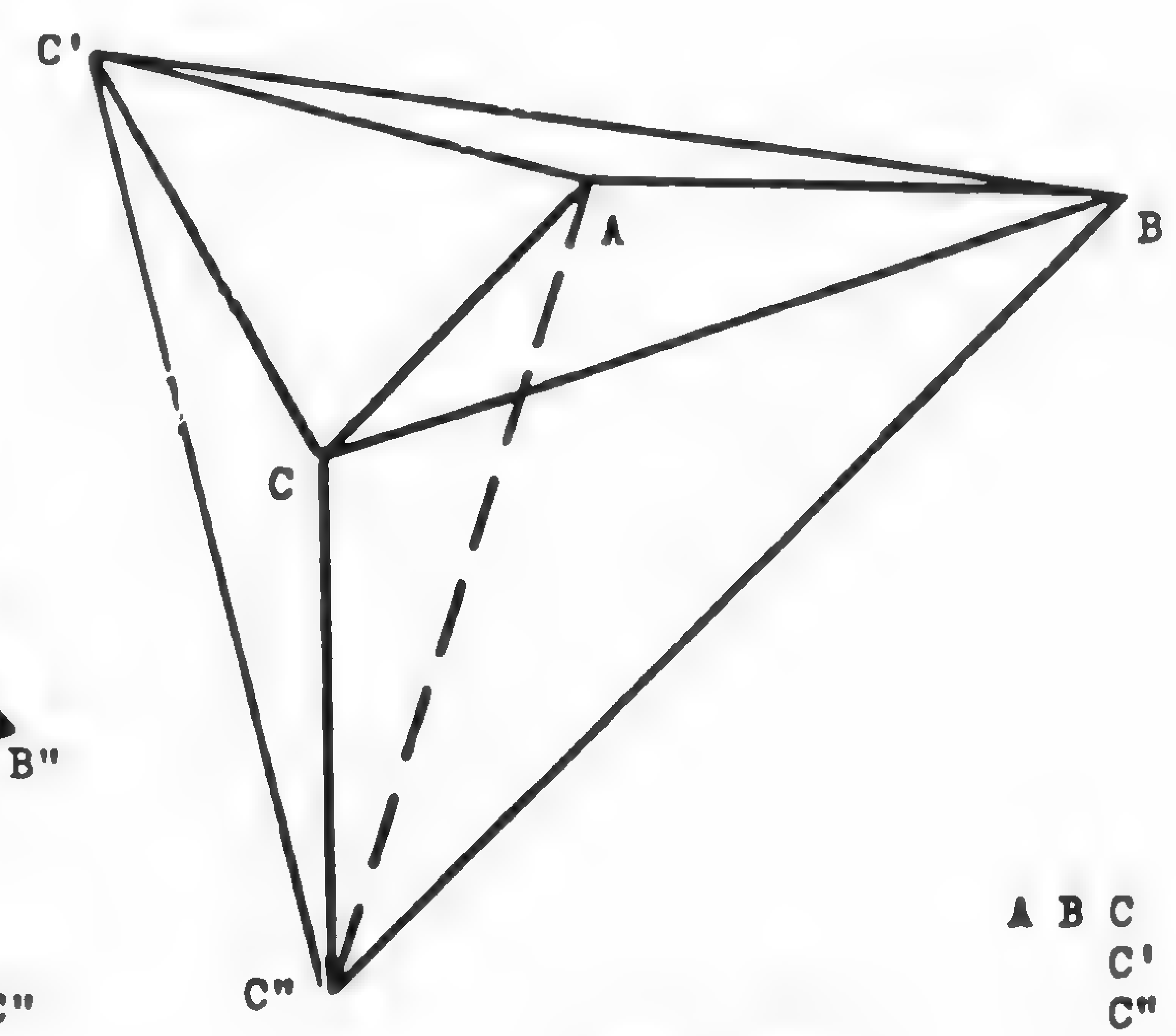
A B C
B'C'
B"C"

Fig. 96b.



A
A'
A"B"C"

Fig. 96aa'



A B C
C'
C"

Fig. 96bb'

We then take the 1st pentahedroid ABCC'C" as a hyperpyramid with vertex at C' and its base as a pyramid C"-ABC with vertex at C" and then re-write the pentahedroid ABCC'C" as the hyperpyramid C'-C"ABC (Fig. 96bb'), and the 2nd pentahedroid AA'A"B"C" as a hyperpyramid with vertex at A' and its base as a pyramid A-A"B"C" with vertex at A by A'-AA"B"C". The 2 pyramids C"-ABC and A-A"B"C" will then have equal bases ABC and A"B"C" respectively, and equal altitudes C"C and AA" respectively, the altitudes being the distance between the planes of these bases. The 2 pyramids C"-ABC and A-A"B"C", are in fact, 2 of the 3 equivalent tetrahedrons into which the triangular-prism ABC-A"B"C" can be divided. The altitudes A'A and C'C of the hyperpyramids A'-AA"B"C" and C'-C"ABC respectively, will be equal to the distance of the plane of the triangle A'B'C' from the hyperplane of the triangular-prism ABC-A"B"C" (see Fig. 96), to which it is parallel. Therefore, the 2 hyperpyramids A'-AA"B"C" and C'-C"ABC are equivalent pentahedroids; that is, the 6 pentahedroids into which the double-prism AA'A"-A"B"C" has been divided are equivalent, and the hypervolume of the double-prism AA'A"-A"B"C" is equal to 6 times the hypervolume of any 1 of these pentahedroids, for example, of the pentahedroid A'AA"B"C", which has among its faces the triangles AA'A" and A"B"C". (Q.E.D)

Theorem 2. The hypervolume of a double-prism is equal to the area of a base of any of the prisms of either set, multiplied by the area of a right directing-polygon of the hypersurface around which this set of prisms extends. (see Fig. 96.)

We can divide the bases of the prisms of the given set into triangles, the given prisms into triangular-prisms, and the double-prism having each set of triangular-prisms for the given set of prisms. The hypersurface around which these prisms extends is the same as for the given double-prism and for all of the double-prisms into which it is divided (Art. 79, Th. and Cor.). Therefore it is only necessary to prove the theorem for double-prisms in which the given set of prisms is a set of triangular-prisms. Further, taking a double-prism with a given set of triangular-prisms for one of its sets of prisms, we can divide the right directing-polygons of the hypersurface around which these prisms extend into triangles. The diagonals which divide one of these directing-polygons into triangles, together with the faces of the hypersurface at their extremities, determine layers which form, with the parts into which they divide the hypersurface, triangular-hypersurfaces, and so divide the double-prism into doubly-triangular prisms. If the theorem is true of doubly-triangular prisms, it is true of any double-prism in which the given set of prisms is a set of triangular-prisms. Therefore, it is only necessary to prove the theorem for doubly-triangular prisms.

Proof: Proceeding as in the proof of Th. 1, we have a pentahedroid ABCC'C" (Fig. 96bb'), 1 of 6 equivalent pentahedroids into which the double-prism AA'A"-A"B"C" can be divided. The volume of the tetrahedron ABCC" is equal to the area of the triangle ABC multiplied by 1/3 of the distance C" from the plane of this triangle, that is, the distance C"C; and the hypervolume of the pentahedroid ABCC'C" is equal to the volume of the tetrahedron ABCC" multiplied by $\frac{1}{4}$ of the distance of C' from the hyperplane of this tetrahedron, the distance being C'C. Thus, the hypervolume of the pentahedroid ABCC'C" is equal to the area of the triangle ABC multiplied by 1/12 of the product of these 2 distances.

Now the plane of the triangle AA'A" (Fig. 96, and in this case, coincides with the directing-triangle AA'A") absolutely-perpendicular to the plane of the triangle ABC at the point A intersects the hypersurface which has this plane for 1 of its faces in a right directing-triangle. The side AA" of the triangle AA'A" measures the distance between the planes of the triangles ABC and A"B"C", and the corresponding altitude A'A of the triangle AA'A" measures the distance of the plane of the triangle A'B'C' from the hyperplane of the planes of the 2 triangles ABC and A"B"C". The area of the triangle AA'A", that is, of the right directing-triangle of the hypersurface which has the plane of the triangle ABC for 1 of its faces, is then equal to $\frac{1}{2}$ of the product of these 2 distances, i.e., the product of the distances being A'A.AA"~~/2~~; and the hypervolume of the pentahedroid ABCC'C" is equal to 1/6 of the hypervolume of the double-prism AA'A"-A"B"C". Therefore, the latter is exactly equal to the product of the areas of the 2 triangles AA'A" and A"B"C", that is, to the area of a base A"B"C" in one set of prisms multiplied by the area of a right-directing-triangle AA'A" of the hypersurface around which this set of prisms extends.

Corollary. The hypervolume of a right double-prism is equal to the product of the areas of its 2 directing-polygons.

96. THEOREMS ON HYPERVOLUMES OF CYLINDRICAL AND CONICAL-HYPERSURFACES.

Theorem 1. The hypervolume of a spherical-hypercylinder is equal to the volume of the base multiplied by the altitude. Its formula is

$$V_4 = \frac{4}{3}\pi r^3 h,$$

r being the radius of the base and h the altitude.

Theorem 2. The hypervolume of a spherical-hypercone is equal to the volume of the base multiplied by $\frac{1}{4}$ of the altitude. Its formula is

$$V_4 = \frac{1}{3}\pi r^3 h,$$

Theorem 3. The hypervolume of a frustrum of a spherical-hypercone is given by the formula

$$V_4 = \frac{1}{4}h(B + B^{2/3}b^{1/3} + B^{1/3}b^{2/3} + b) = \frac{1}{3}\pi h(R^3 + R^2r + Rr^2 + r),$$

where the upper-base = $4/3\pi r^3$ (volume of a sphere with radius r) and lower-base = $4/3\pi R^3$ (volume of a sphere with radius R). (Fig. 97.)

Compare the relationship of this formula to the one for the hypervolume of the frustrum of a hyperpyramid. The proof is similar to that for the frustrum of a hyperpyramid (see Art. 94).

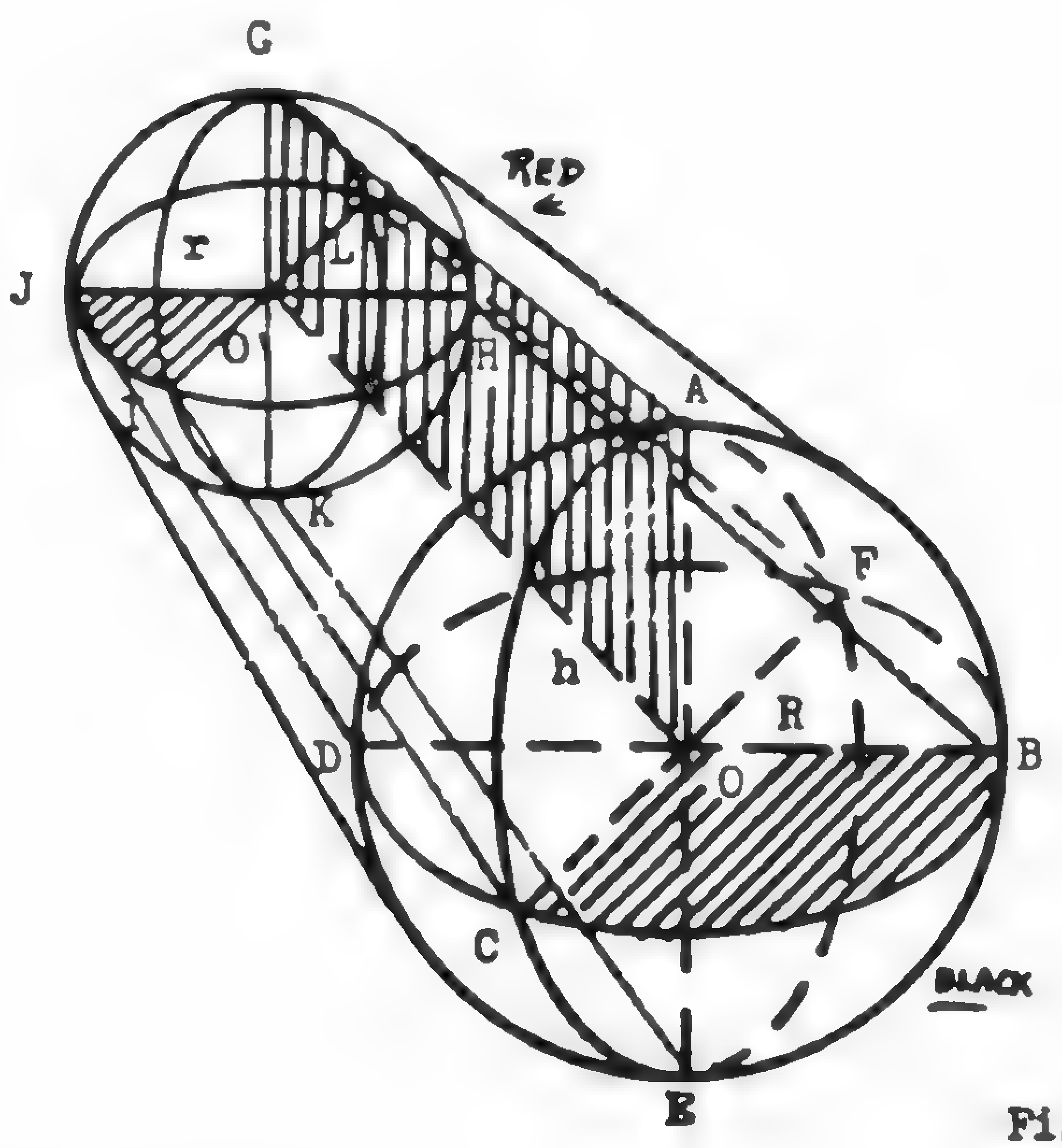


Fig. 97.

The formula can also be written as

$$V_4 = \frac{1}{3}\pi h(R + r)(R^2 + r^2).$$

Theorem 4. The hypervolume of a right prism-cylinder with directing-circle is equal to the product of the areas of the directing-polygon and directing-circle. Its formula is

$$V_4 = \pi r^2 A,$$

A being the area of the directing-polygon and r being the radius of the directing-circle.

Theorem 5. The hypervolume of a cylinder of double-revolution is equal to the product of the areas of its directing-circles. Its formula is

$$V_4 = \pi^2 r^2 r'^2.$$

97. THEOREMS ON VOLUME AND HYPERVOLUME OF A HYPERSPHERE.

Theorem. The volume of a hypersphere is given by the formula

$$V_4 = 2\pi^2 r^3,$$

r being the radius.

In the next chapter we shall use the method of the calculus to find the volume of a hypersphere.

Corollary. The volume of a cylinder of double-revolution circumscribed about a hypersphere is twice the volume of the hypersphere.

Theorem. The hypervolume of a hypersphere is equal to its volume multiplied by $\frac{1}{4}$ of its radius.

Corollary. The hypervolume of the hypersphere is given by the formula

$$V_4 = \tfrac{1}{2}\pi^2 r^4.$$

In the next chapter we shall use the method of the calculus to find the hypervolume of a hypersphere.

Corollary.2. The hypervolume of a hypersphere is equal to $\frac{1}{2}$ of the hypervolume of the circumscribed double-cylinder, and twice the hypervolume of the inscribed double-cylinder, and twice the hypervolume of the inscribed double-cylinder with equal radii. It is equal to the hypervolume of any inscribed double-cylinder plus the hypervolumes of 2 hyperspheres whose radii are the radii of the double-cylinder.

The student can consult Manning's Geometry of Four Dimensions for proofs on the volume and hypervolume of a hypersphere (see pp. 267-270, and pp. 285-287). Manning uses the theory of limits to determine the volume and hypervolume of a hypersphere.

98. CAVALIERI'S THEOREM. A very useful method for calculating the hypervolumes of certain irregular hypersolids is the extension of Cavaleri's theorem to include hypersolids.

Cavalieri's Theorem (for hypersolids). If in 2 hypersolids of equal heights the sections made by hyperplanes parallel to and at the same distance from their respective hyperplane bases are always equal, the hypervolumes of the hypersolids are equal.

Many theorems on hypervolumes can be proved by making use of Cavaleri's theorem and Th. 3 of Art. 93.

In the next chapter we shall apply the hypersolid-geometry to Descartes' Analytic-geometry, that is, Cartesian-coordinate-geometry.

In this short-treatise we have covered the 'basics' of hypersolid-geometry making use of theorems, proofs, and graphics together with new-definitions and symbol-notations as required. The axioms of both the Euclidean and Non-Euclidean geometries are assumed.

Many new insights of the hypersolid-geometry are derived from the graphics, however, the author of this treatise realizes that the hyperspace-graphics can be developed much further, many new applications will lead to further discoveries in many branches of pure-mathematics, engineering, and physics.

CHAPTER IX

EXCURSUS

I. HYPERSOLID-ANALYTIC-GEOMETRY

99. APPLICATIONS. In this chapter we shall combine the hypersolid-geometry with the analytic-geometry of 4 dimensions via a rectangular-coordinate-system, then make a few applications of the hypersolid-analytic-geometry. In the next section we shall make use of the calculus to find the volume and hypervolume of a hypersphere and also making use of the hyperspace-graphics as developed in this text.

100. RECTANGULAR CARTESIAN-COORDINATES. In hypersolid-analytic-geometry one method of locating a point in hyperspace is in terms of its perpendicular distance from 4 mutually-perpendicular hyperplanes. These hyperplanes are called COORDINATE-HYPERPLANES, and the 4 perpendicular distances are called COORDINATES OF THE POINT.

The lines of intersection of these coordinate-hyperplanes are the 4 axes OX, OY, OZ, and OW, called COORDINATE-AXES, with the positive-directions shown by arrows (see Fig. 98). The coordinate-hyperplanes divide all hyperspace into 16 parts called HEXADEKANTS as follows: Number 1 is the PRINCIPAL-HEXADEKANT whose bounding-edges are the positive-directions of the 4 coordinate-axes; then numbers 2 to 8 lie above the xyz-hyperplane in a counterclockwise-direction about OW, that is, around the zw-plane as axis-plane which contains OW. Numbers 9 to 16 lie below the xyz-hyperplane, number 9 lying under number 1.

In the figure the distances QP, RP, SP, and NP are respectively the x, y, z, and w coordinates of the point P, and the point P may be denoted by the 4-tuple (x,y,z,w), or P(x,y,z,w).

The distance OP of the point P from the origin O is

$$OP = (\overline{ON}^2 + \overline{NP}^2)^{\frac{1}{2}} = (\overline{OM}^2 + \overline{MN}^2 + \overline{NP}^2)^{\frac{1}{2}} = (\overline{OL}^2 + \overline{LM}^2 + \overline{MN}^2 + \overline{NP}^2)^{\frac{1}{2}} = (x^2 + y^2 + z^2 + w^2)^{\frac{1}{2}}.$$

Letting $OP = \rho$, then $\rho^2 = x^2 + y^2 + z^2 + w^2$.

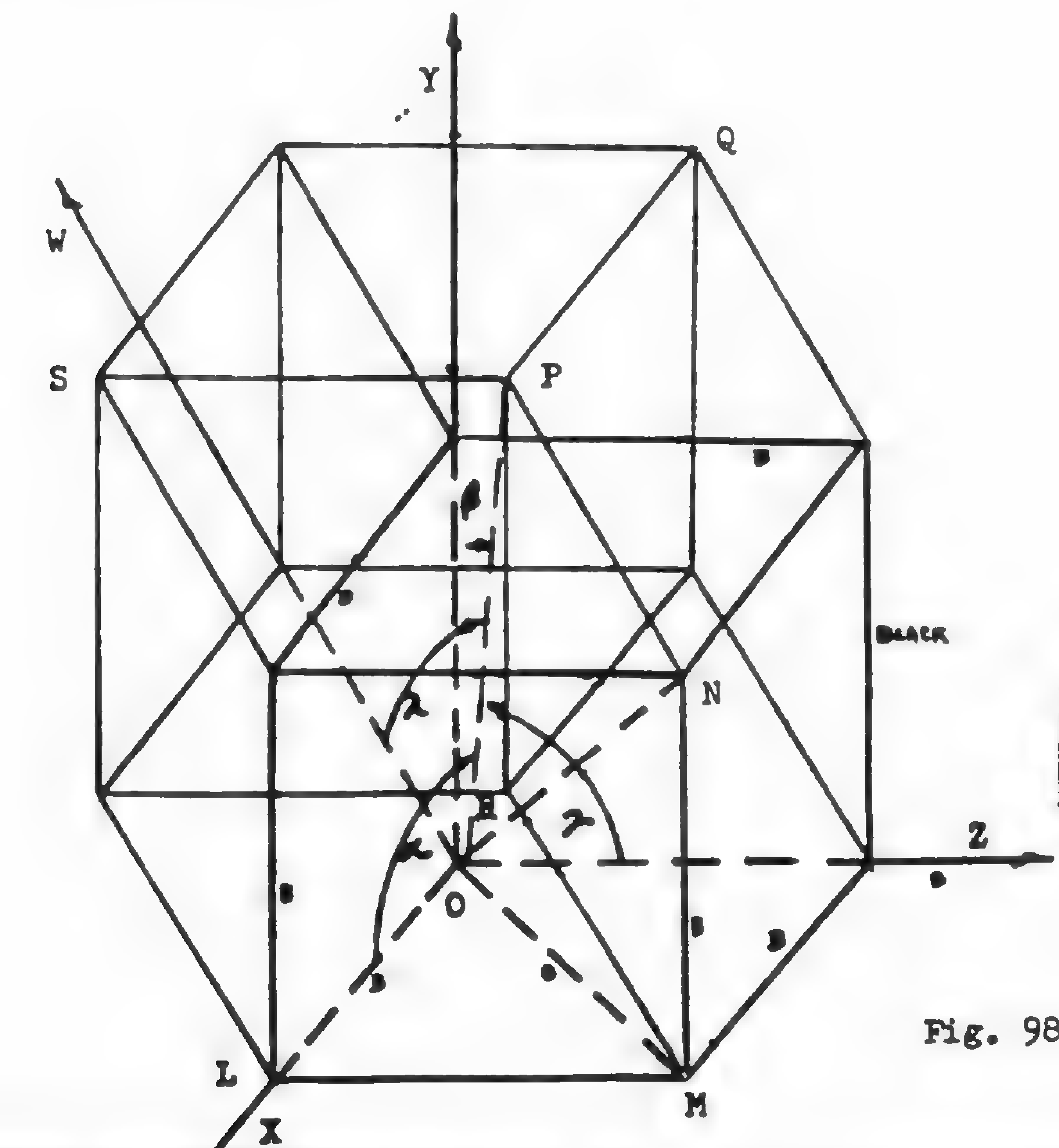


Fig. 98.

101. DIRECTIONAL-ANGLES AND DIRECTIONAL-COSINES. Let the angles between OP and OX, OY, OZ, OW be respectively $\alpha, \beta, \gamma, \lambda$. Then,

$$x = \rho\cos\alpha, \quad y = \rho\cos\beta, \quad z = \rho\cos\gamma, \quad w = \rho\cos\lambda .$$

Squaring these relations and adding,

$$x^2 + y^2 + z^2 + w^2 = \rho^2\cos^2\alpha + \rho^2\cos^2\beta + \rho^2\cos^2\gamma + \rho^2\cos^2\lambda, \text{ or}$$

$$1 = \cos^2\alpha + \cos^2\beta + \cos^2\gamma + \cos^2\lambda .$$

We have also the relations $\cos\ = x/\rho$, $\cos\ = y/\rho$, $\cos\ = z/\rho$, $\cos\ = w/\rho$, or

$$\cos\alpha = \frac{x}{(x^2 + y^2 + z^2 + w^2)^{\frac{1}{2}}}, \quad \cos\beta = \frac{y}{(x^2 + y^2 + z^2 + w^2)^{\frac{1}{2}}},$$

$$\cos\gamma = \frac{z}{(x^2 + y^2 + z^2 + w^2)^{\frac{1}{2}}}, \quad \cos\lambda = \frac{w}{(x^2 + y^2 + z^2 + w^2)^{\frac{1}{2}}} .$$

The angles $\alpha, \beta, \gamma, \lambda$ of the line OP are called the DIRECTION-ANGLES of OP, and the cosines of these angles are called the DIRECTION-COSINES of OP.

If a line does not pass through the origin O, then its direction-angles $\alpha, \beta, \gamma, \lambda$ are the angles between the axes and a line drawn through O parallel to the given line and having the same direction.

102. DIRECTION-NUMBERS. Any 4 numbers a, b, c, and d proportional to the direction-cosines of a line are called DIRECTION-NUMBERS of the line. To find the direction-cosines of a line whose direction-numbers a, b, c, and d are known, divide the numbers by $\pm(a^2 + b^2 + c^2 + d^2)^{\frac{1}{2}}$. Use the sign in front of the square-root expression which will will cause the resulting direction-cosines to have the proper-sign.

103. DISTANCE BETWEEN 2 POINTS. The distance between any 2 points $P_1(x_1, y_1, z_1, w_1)$ and $P_2(x_2, y_2, z_2, w_2)$ is

$$d = ((x_2 - x_1)^2 + (y_2 - y_1)^2 + (z_2 - z_1)^2 + (w_2 - w_1)^2)^{\frac{1}{2}}.$$

104. DIRECTION OF A LINE. The direction-cosines of P_1P_2 are

$$\cos\alpha = \frac{x_2 - x_1}{((x_2 - x_1)^2 + \dots + (w_2 - w_1)^2)^{\frac{1}{2}}}, \dots, \cos\lambda = \frac{w_2 - w_1}{((x_2 - x_1)^2 + \dots + (w_2 - w_1))^{\frac{1}{2}}}.$$

II HYPERPLANES

105. NORMAL-FORM. INTERCEPT-FORM. DISTANCE FROM A POINT TO A HYPERPLANE. Every hyperplane is represented by an equation of the 1st degree in 1 or more of the variables x, y, z, w. The converse statement is also true. Every equation of the 1st degree in 1 or more of the variables x, y, z, w represents a hyperplane.

The NORMAL-FORM of the equation of a hyperplane is

$$x\cos\alpha + y\cos\beta + z\cos\gamma + w\cos\lambda - \rho = 0,$$

where ρ is the perpendicular distance from the origin to the hyperplane and $\alpha, \beta, \gamma, \lambda$ are the direction-angles of that perpendicular.

The normal-form of the equation of the hyperplane $Ax + By + Cz + Dw + E = 0$ is

$$\frac{Ax + By + Cz + Dw + E}{\pm(A^2 + B^2 + C^2 + D^2)^{\frac{1}{2}}} = 0,$$

the sign in front of the square-root expression being taken opposite to that of E so that

the normal-distance p shall be positive.

The INTERCEPT-FORM of the equation of a hyperplane is $x/a + y/b + z/c + w/d = 1$, where a, b, c, d are the x-, y-, z-, and w-intercepts respectively.

THE DISTANCE OF A POINT FROM A HYPERPLANE. The perpendicular-distance between a point $(x_1, \dots, w_1)$ and a hyperplane $Ax + By + Cz + Dw + E = 0$ is

$$d = \left| \frac{Ax_1 + \dots + Dw_1 + E}{(A^2 + \dots + D^2)^{\frac{1}{2}}} \right| .$$

Problem. Discuss the locus of the equation $x + y + z + w = 1$. (Fig. 99.)

Since the equation is of the 1st degree it represents a hyperplane. The direction-numbers of the normal to the hyperplane are 1,1,1, and 1. The direction-cosines of the normal are $\cos\alpha = \frac{1}{2}$, $\cos\beta = \frac{1}{2}$, $\cos\gamma = \frac{1}{2}$, and $\cos\lambda = \frac{1}{2}$.
The intercepts on the axes are (1,0,0,0), (0,1,0,0), (0,0,1,0), and (0,0,0,1).

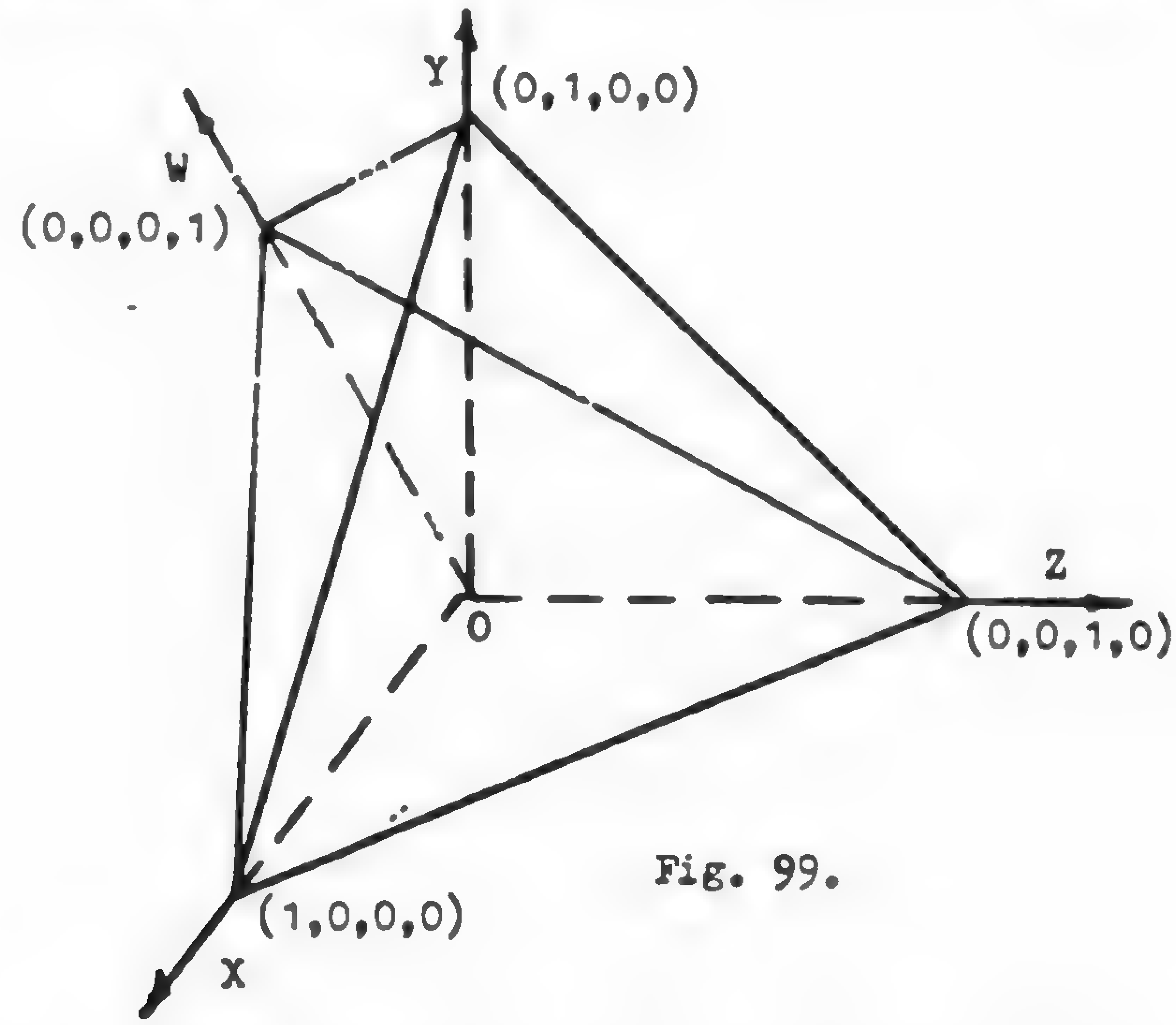


Fig. 99.

The planes in which the hyperplane intersects the coordinate-hyperplanes are called the SPREADS of the hyperplane. To find the equations of the spreads: In the xyz-hyperplane, $w = 0$; hence the equation of this spread is $x + y + z = 1$. The equations of the other spreads are found by setting x, y, and z = 0, respectively, giving us the equations of the spreads $y + z + w = 1$, $x + z + w = 1$, and $x + y + w = 1$.
The lines in which the plane of a spread intersects the coordinate-hyperplanes are called the * ~~TRACES OF THE PLANE OF THE SPREAD~~. To find the equations of the traces: In the xz-plane, $y = 0$, $w = 0$; hence the equation of this trace is $x + y = 1$. The equations of the other traces are found by setting 2 of the variables in the equation $x + y + z + w = 1$ equal to 0; all-together, we shall have 6 traces.

The 4 intercepts, 6 traces, and 4 spreads are shown in the figure above.

To find the length of the normal, that is, the distance from the origin to the hyperplane:

$$d = \left| \frac{Ax_1 + By_1 + Cz_1 + Dw_1 + E}{\pm(A^2 + B^2 + C^2 + D^2)^{\frac{1}{2}}} \right| . \qquad d = \left| \frac{1(0) + 1(0) + 1(0) + 1(0) - 1}{(4)^{\frac{1}{2}}} \right| = \frac{1}{2}.$$

106. QUADRATIC-HYPERSURFACES. The hypersurface defined by an equation of the 2nd degree

* the TRACES OF THE PLANES OF THE SPREAD.", should read-as, "The lines in which the hyperplane intersects the coordinate-planes are called the TRACES of the hyperplane.".

in 4 variables is called a QUADRATIC-HYPERSURFACE. Any hyperplane-section of a quadratic-hypersurface is a hyperconic or a limiting-form of a hyperconic. We shall list a few equations of some of the quadratic-hypersurfaces:

THE HYPERSPHERE. $x^2 + y^2 + z^2 + w^2 = a^2$, a being the radius and (0,0,0,0) its center.

THE HYPERELLIPSOID. $\frac{x^2}{a^2} + \frac{y^2}{b^2} + \frac{z^2}{c^2} + \frac{w^2}{d^2} = 1$, the center being the origin.

HYPER-HYPERBOLOID OF 1 MATRESS. $\frac{x^2}{a^2} + \frac{y^2}{b^2} + \frac{z^2}{c^2} - \frac{w^2}{d^2} = 1$

HYPER-HYPERBOLID OF 2 MATRESSES. $\frac{x^2}{a^2} - \frac{y^2}{b^2} - \frac{z^2}{c^2} - \frac{w^2}{d^2} = 1$

CONICAL-HYPERSURFACE. $\frac{x^2}{a^2} + \frac{y^2}{b^2} + \frac{z^2}{c^2} - \frac{w^2}{d^2} = 0$

HYPERCYLINDRICAL-HYPERSURFACE. $\frac{x^2}{a^2} + \frac{y^2}{b^2} + \frac{z^2}{c^2} = 1$, with $w = 0$, the equation becomes the directing-surface, or directing-ellipsoid.

III. THE VOLUME AND HYPERVOLUME OF A HYPERSPHERE.

107. HYPERVOLUME OF HYPERSOLIDS WITH KNOWN CROSS-SECTIONS. VOLUME AND HYPERVOLUME OF A HYPERSPHERE. If the volume of a cross-section of a hypersolid make by a hyperplane perpendicular to 1 of the axes, say the X-axis, and at a distance x from the origin, can be expressed as a function A(x) of x, the hypervolume of the hypersolid is given by the formula

$$(1) \qquad V_4 = \int_a^b A(x)dx.$$

In other words, if we consider the sections of a hypersolid made by parallel hyperplanes perpendicular to the X-axis, then if it is possible to write down the volume of each section in terms of its distance from some fixed-point on the X-axis, say the origin O, then the hypervolume of the hypersolid can be determined. For the volume of a cross-section at a given distance x is known to be a function of x, say A(x), then the hypervolume-element is A(x)dx. Hence the hypervolume from x = a to x = b is given by (1).

Hypersolids of revolution are a special-case of this.

In Fig. 100 (here we use an opposite-order of a rectangular-system by interchanging the axes in the xyz-hyperplane), if we take a hyperplane perpendicular to the X-axis at a given distance from the origin O (the center of a hypersphere H), then the volume of the cross-section will be an 1/8-sphere of H. If we denote by w the radius of this 1/8-sphere, then its volume is $1/6\,\pi w^3$. But it can be represented in terms of w since $w^2 + x^2 = r^2$ (x and w lie in the XZ-plane and are at right-angles to each other, r being the radius-vector, then the triangle formed of w, x, and r will be a right-triangle, with r being the hypothemuse of this triangle). The volume is then $1/6\,\pi(r^2 - x^2)^{3/2}$. The hypervolume-element becomes $1/6\,\pi(r^2 - x^2)^{3/2}dx$. The hypersphere H being symmetric, we can write its hypervolume as

$$V_4 = 16\int_0^r \pi\,\frac{1}{6}(r^2 - x^2)^{3/2}dx,$$

it being understood that the limits of integration are from x = 0 to x = r and the

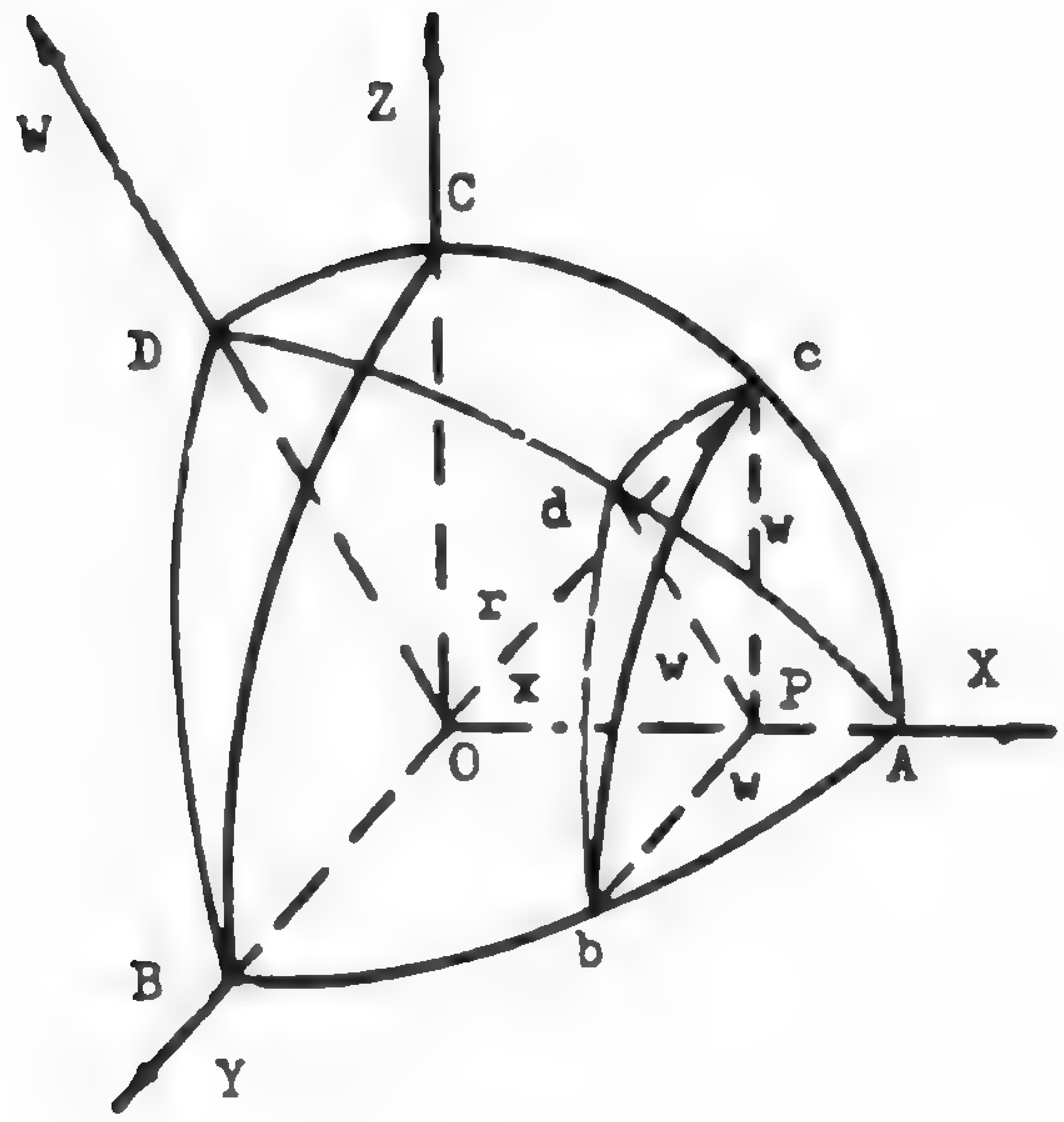


Fig. 100.

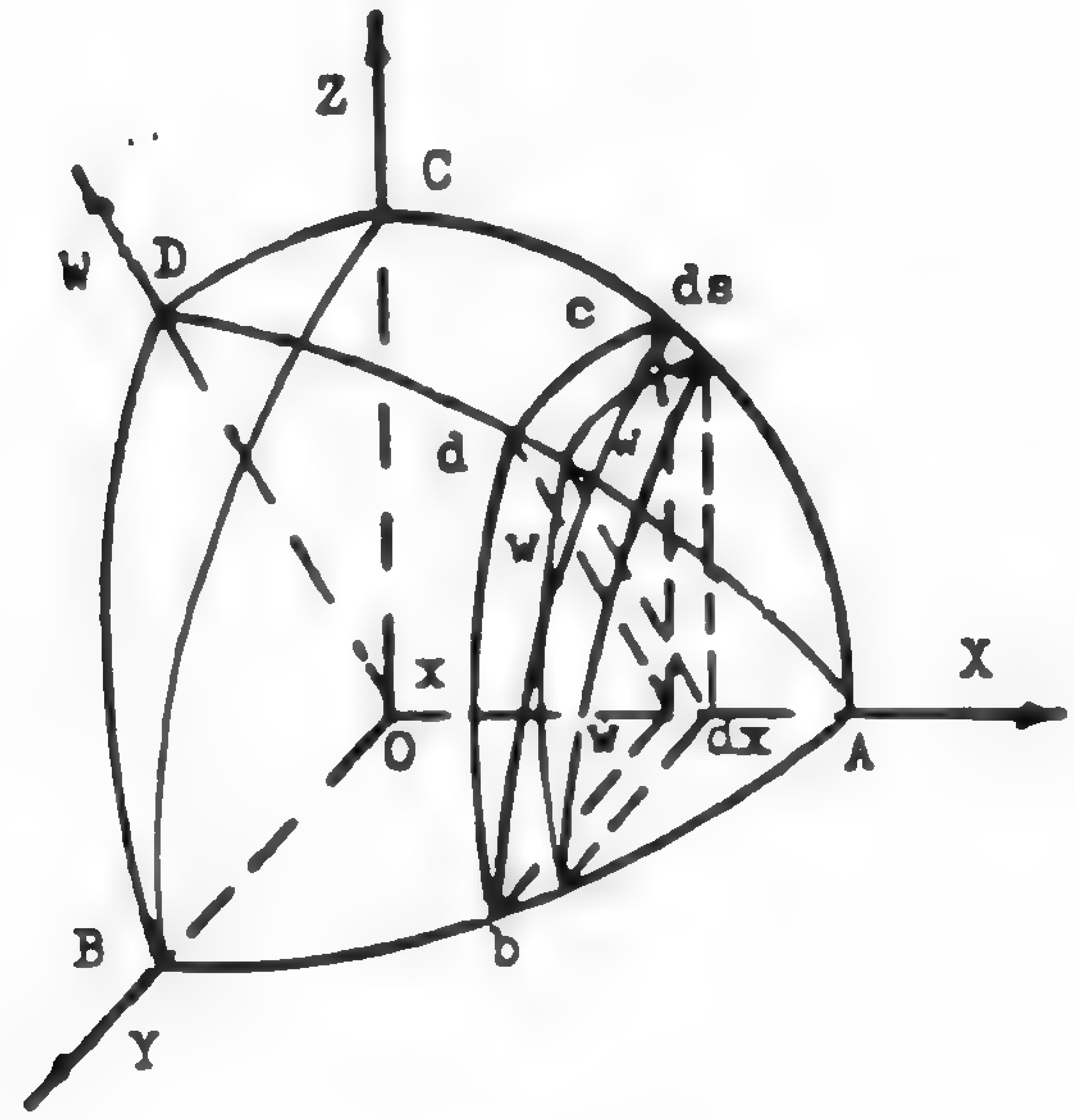


Fig. 101.

volume-portion represents only an 1/8-sphere, and therefore we must multiply the definite-integral by 2.8, or 16. We then have

$$V_4 = \frac{8}{3}\pi\int_0^r (r^2 - x^2)^{3/2}dx$$
$$= \frac{8}{3}\pi\left[\frac{x}{4}(r^2 - x^2)^{3/2} + \frac{3r^2x}{8}(r^2 - x^2)^{\frac{1}{2}} + \frac{3r^4}{8}\sin^{-1}\frac{x}{r}\right]_0^r$$
$$= \frac{8}{3}\pi\left(\frac{3r^4}{8}\cdot\frac{\pi}{2}\right) = \frac{\pi^2r^4}{2}.$$

We can determine the volume of a hypersphere H by considering the area of the cross-section of the surface of a sphere at a given distance from the origin, the sphere being that as explained above. The area of the surface of the 1/8-sphere is $\pi w^2/2$, and we take for dx the element of arc ds as shown in Fig. 101, that is, ds is a function of (dw, dx) with $ds = (1 + w'^2)dx$, where $w' = dw/dx$. The volume-element becomes $\pi w^2/2\ ds$. The volume of H is given by

$$V_4 = 16\int_0^r \frac{\pi w^2\ ds}{2} = 8\pi\int_0^r w^2ds$$
$$= 8\pi\int_0^r w^2(1 + (dw/dx)^2)^{\frac{1}{2}}dx = 8\pi\int_0^r w^2(w^2 + x^2/w^2)^{\frac{1}{2}}dx$$
$$= 8\pi\int_0^r w^2.r/w\ dx = 8\pi r\int_0^r w\ dx = 8\pi r\int_0^r (r^2 - x^2)^{\frac{1}{2}}dx$$
$$= 8\pi r\left[\frac{x}{2}(r^2 - x^2)^{\frac{1}{2}} + \frac{x^2}{2}\sin^{-1}\frac{x}{r}\right]_0^r = 8\pi r\left(\frac{r^2}{2}\cdot\frac{\pi}{2}\right) = 2\pi^2r^3.$$

ADDENDUM

MULTIDIMENSIONALITY AND THE MATTER-SPECTRUM

A few gifted individuals have speculated on the possibility that 'space' as-we-know-of -it may have more than 3 spacial-dimensions. Many present-day 'thinkers' have assumed that any higher-space must exist in other than spacial-coordinates, somewhat like the limited relativistic-physics of today that makes use of 3 spacial-coordinates and 1 time-coordinate. Others speak about an 'ether' as the substrate from which new-dimensions might spring forth, whatever-that-may-mean. Still others consider 'mystic-experiences' other-dimensional and beyond what we call 'physical-matter'. Some seem to detest the idea of a matter-universe and say that we have 'souls' independent of physical-matter, whatever that may mean in an operational-sense. A large number of individuals 'think' that we are welded to a physical-universe existence as-if it were the only 'reality' possible. We see a split in 'thinking' (thought-process) between 2 opposite viewpoints where 'mind' is something different from 'physical-matter', and we end up in a vicious-circle in logical-reasoning like: 'mind or matter'? We could go on-and-on with this kind of primitive-logic and achieve little understanding of the laws which govern 'mind-matter' processes. Using a 2-valued logic system of reasoning will lead us to no viable results that can be experimentally verified. Besides, the 2-valued logic methods of 'reasoning' are non-operational in larger frames of reference; that is, the law of opposites does not apply except in very limited-cases, and therefore, when multi-viewpoints are taken into consideration in formulating theories from data (classification of events), different conclusions will be inferred about the laws which govern reality-gestalts of energy-systems existing in multidimensional-space.

The 1st crack in the 2-valued logic scheme came with the discovery of anti-matter. If W say, "Is it matter or not matter?" Then by the law of the excluded-middle, anti-matter cannot be matter, but it so happens that anti-matter and matter are but different manifestations of 'energy', and therefore anti-matter must be an aspect of matter. But this leads to the absurd conclusion that anti-matter is matter. The vicious-circle of the 2-valued logic then becomes resolved by some kind of 'duality-principle' leading to further confusion of the laws which govern energy-gestalts of what's what there on non-verbal-levels of 'reality'.

Now if we make use of an infinite-valued system of logic, then new-gestalt-patterns of awareness become apparent, thus enabling us to generate higher-order OPERATIONAL-concepts leading to new-ways of forming reality-frameworks. What we are interested in mainly, is the translation of ideas and concepts into FUNCTIONAL-models in what we agree to call 'new-realities', the degree of agreement being determined by the degree of 'workability' of the functional-models.

To make any progress at all in the unsolved-problem of 'mind or matter' we must step outside of our 3-space cell, there being no other way possible. We begin by assuming that our so called CONSCIOUSNESS exist independent of 'matter' as-we-know-of-it—we are outside looking in, as the saying goes. We have physical-senses tuned-in to perceive the 'effects' of happenings within physical-matter. But to get outside of our so called 3-space cell, we must make use of what has been called the <u>inner-senses</u> existing beyond the limits of so called physical-matter. The inner-senses, then, enable us to perceive inner-realities existing below the range of 'objects' composed of physical-matter. But no 'matter' how we try to evade the issue of "matter", we still have 1 common-denominator that pervades all existence, namely, 'energy'. But the types of energy are infinite. We must begin somewhere, and we begin by assuming that a PRIMAL-ENERGY exist and acts as the source from which all the dimensions of existence spring forth, and further, that the primal-energy is assumed to be NEUTRAL. Now a part of the primal-energy goes to create ENERGY-ESSENCE-PERSONALITIES OF CONSCIOUSNESS independent of matter, time, space, and so forth, as-we-know-of-it. From this we deduce that each energy-essence-personality creates FORM or OTHERWISE and evolves the multispaces of dimensions to manifest the ideas created by its psychic-idea-projections, and by feedback of the so called projected-dimensional-materializations enables it to create new and better reality-gestalts ad infinitum. But this leads to the conclusion that 'time' is an ILLUSION and 'matter' NOT

really 'solid' after all.

We have agreed that a light-spectrum exist from theories and experiments that confirm the results of many 'observers'. We then postulate that a MATTER-SPECTRUM likewise exist. We can distinguish the different 'phase-spaces' of the matter-spectrum by the INTENSITY of 'matter' in which a given space-time frame exist, or again by other hyperspace-coordinates in which a given intensity-mass does not have a 'time-element' as-we-know-of-it—it being understood that the properties of pure-energy are unlimited, and so for a certain intensity-mass, its so called matter-properties which we call its energy-gestalt will be different from our 3-space 'viewpoints'. Each intensity-mass, then, will have specific-laws in such a phase-space. Further, the energy-essence-personality projecting such an intensity-mass will create <u>outer-senses</u> to manipulate the intensity-mass within its range. For consciousness, then, the common-denominator will be the inner-senses which SPAN the different dimensions of the intensity-masses.

But we still have not made use of operational-methods to manipulate the intensity-masses within the matter-spectrum. Now in a matter-spectrum energies will interact. The law of 'entropy' will no longer hold valid within an open-ended energy-system. So we postulate a system of COORDINATE-POINTS that allow energies from different 'continuua' to flux from 1 local-energy-system to another. These coordinate-points could be called PSYCHIC-GENERATORS that propell the atoms into different energy-gestalts via polarity-groupings of different kinds. The coordinate-points would be something like 'BLACK-WHITE HOLES' in hyperspace allowing energies to move from 1 intensity-mass system to another. The coordinate-points would be powerhouses containing traces of pure-energy. We can consider that in a 4-space framework, we would have 4 absolute-coordinate-points generating ultra-energy potentials and an indefinite-number of subcoordiante-points generating lesser traces of pure-energy.

It is then 1 step further in which to derive an operational-concept for the matter-spectrum. We then postulate an ENERGY-POINT initiating the 1st-stage in the formation of an energy-gestalt giving rise to a certain intensity-mass. The consciousness (inner-self) using intense EMOTIONAL-ENERGY projects the energy-point into existence, the INTENSITY of the psychic-idea-image determining the 'density' of the CORE about the energy-point being created by consciousness. The emergence of the energy-point then gathers about itself a field of ELECTROMAGNETIC-ENERGY with the energy-point inside—like a triangle-effect, then transforms into a spinning freewheeling electromagnetic-field. We call this spinning freewheeling electromagnetic-field with the energy-point inside an EE-UNIT. Now since consciousness projects these EE-units into existence, then because of the difference in the intensities of the EE-units, POLARITIES will come about, meaning, different EE-units will polarize into energy-gestalts of various sizes and forms. The polarized EE-units PULSATE, EXPAND or CONTRACT, in an unlimited-number of ways. When the polarized EE-units attain a certain intensity they then break through into physical-matter by compression created by polarization-groupings of the EE-units. The EE-units can also depolarize and form new gestalt-groupings. In order to bring the EE-units into the higher-intensity-mass ranges, or lower for that matter, the EE-units are projected through the coordinate-points at enormous-velocities, that is, the coordinate-points become BLACK-HOLES, then at some point turn inside-out and become WHITE-HOLES again, with the EE-units being projected back again into a certain intensity-range which then generate polarity-groupings bringing about density (pressure) and break through into a given intensity-mass within the range of the matter-spectrum in which our physical-bodies exist in; it being understood that the consciousness (inner-self) will project other EE-units into other intensity-ranges in which the intensity-masses will lie in other ranges of the matter-spectrum, and having PROBABLE-SELVES created by the inner-self of which we are part of in our 3-space cell, and so forth.

We can, then, by use of the inner-senses of 'consciousness' create an idea, project it into the range of the matter-spectrum in which our physical-bodies lie, that is, for a given intensity-mass. Further, it being understood that the physical-body and its outer-senses are merely thought-projections of the inner-self (consciousness) that manifest a certain intensity-mass in order that it can manipulate the given intensity-mass lying in a given phase-space. The physical-body pulses at enourmous cycles/sec within the given intensity-mass field in which it exist in, and so forth. By feedback, the consciousness

then creates 'physical-objects', which we call OBJECT-SYMBOLS, by which it realizes its own creativity in the manipulation of the energy-gestalts existing within the different ranges of the matter-spectrum. The consciousness, being immortal, creates endless dimensions to gain personal-experience in the development of its creative-abilities. The source of its energy-essence being assumed to come from the 'nameless-one' called "ALL THAT IS"—what we would call primal-energy; the primal-energy being NEUTRAL, then, whatever the energy-essence-personality (entity or consciousness) projects will become its own creations. Since the energy-essence-personality, being a multidimensional-personality, must then have multiple-selves existing in the simultaneous-dimensions of the matter-spectrum from which the inner-self develops its potential—the past and future can then be changed at will by one's consciousness, or what we have called the energy-essence-personality or entity.

In some energy-systems used by consciousness, that is, in some other dimensions having degrees of parallelism with the intensity-mass of our energy-gestalt-system, 'time' would have no-meaning whatsoever. Events would be simultaneous in such pyramid-energy-gestalt-systems. In our intensity-mass system we register 'time' in a linear-fashion as the pulse-rate between nerve-impulses—in fact, really, the time-sequence seems as-if it becomes more like CYCLES OF ACTION, or action-counteraction, and which may be called time-circles.

We could postulate the idea of PSI-TIME via use of the inner-senses enabling us to leap the so called physical-time barrier created by the intensity-mass system in which our physical-bodies exist in. With psi-time, then, we could move into the past or future, in no-time

The MEMORY-BANKS of consciousness seems to be made-up of a VORTEX OF ELECTROMAGNETICALLY-COILED EE-units that link the non-physical-mind and physical-brain into pyramid-energy-gestalts. The coiled-electromagnetic-field of consciousness contains the memory-banks existing simultaneously in all the different dimensions. The consciousness then has the abilities to ponder over PROBABLE-REALITIES AND NEW-DIMENSIONS OF EXPERIENCE. The invisible primal-energy-matrix then becomes the source from which consciousness draws its energy to create the probable-dimensions polyrealities which it brings into existence as such.

Now many of the operational-concepts and ideas described above can be found in the works of Jane Roberts: SETH SPEAKS and the SETH MATERIAL, Prentice Hall. This undoubtedly is the greatest material yet on concepts relating to 'matter-construction' and the 'immortality' of the individual as well as on concepts pertaining to the 'multidimensional-personality'. The clues given in the Seth Material are enough to create a number of psi-sciences pertaining to the THOUGHT-LAWS governing <u>some</u> matter-constructions

Now then, we come to ways in which we can attempt to begin the processes involved in the manipulation of the matter-spectrum. Where? We ponder over it and come to a probable-conclusion that the starting-point that opens the hidden-door to the higher-dimensions lies with the GREAT PYRAMID OF EGYPT; that is, the energy-properties of the Great Pyramid.

The works of G. Pat Flanagan and his theory of pyramid-energy as a vortex-energy generated by the cross-action of an electrostatic-field and magnetic-field, or what he calls the TRANSVERSE-FIELD EFFECT generated by superimposing the electrostatic-field with a magnetic-field. The Cheops pyramid, then, becomes a SHAPED-ELECTROSTATIC-FIELD cross-acting with a MAGNETIC-FIELD. The vortex-theory of the atom as theorized by Flanagan has been verified by an experiment with the radiation-microscope—fast-spinning TAURUS-like RINGS of particles and their BANDS OF ENERGY within the nuceli of pure-iron. For the works of Flanagan see 'Pyramid Power', published by De Vorss & Co.

Since vortex-energies are involved, that is, generated by geometrical-forms like the pyramid or cone, then we can conjecture that GEOMETRICAL-FORMS are the means whereby the different dimensions can be jumped from 1 intensity-mass system to another, where the combinations of the geometric-forms generate and focus the so called vortex-energies. The vortex-atom then becomes the 1st-stage in our manipulation of the matter-spectrum. By going further, we could postulate a 4-space, 5-space,..., n-space 'atom'. The multidimensional-atom would pulsate from 1 intensity-mass system to another by altering the enveloping electromagnetic-field about it, meaning, changing the polarity-bands of the enveloping vortex-energy, and so forth. The <u>atoms</u> that we see in our intensity-mass range pulsate so fast that it seems as-if the atom were continuous, but this is not a property of energy, each bit of energy within an energy-gestalt pulsates, and continuity

as-we-know-of-it, is an illusion of the outer-senses. One's consciousness has pyramid-energy-gestalts of atoms pulsating in ranges that lie above and below the intensity-mass of our so called universe of physical-matter. Period. See the Seth Book by Jane Roberts, p-104, for an interesting example of a 'supertable' having an intensity-mass-gradient assigned to it--x-chairs of different intensity-masses.

JOHN P. BOYLE'S OCCULT-ILLUMINATOR (Mystic-Pyramid)

One interesting 'light' has been shed on the properties of TIME-DISTORTION by the 'occult-illuminator' invented by John P. Boyle of Astral Research Co., PO Box 583-A, Detroit, Michigan 48232. The occult-illuminator mades use of psi-time in visualizing the projected thought-forms on the mirror-reflector lying at the base of the pyramid (Cheops Pyramid modified into a thought-form projector). Using psi-time, the inner-eye makes use of the inner-senses and projects the 'weak' psychic-idea-images onto the mirror -reflector lying at the base of the occult-illuminator, that is, 'consciousness' generates EE-units into a certain psi-intensity-range which become manifested as materialized thought-forms. We could call these transformed EE-units PSITRONS as postulated by the noted English mathematician Adrian Dobbs.

It seems as-if the vortex-energies of the occult-illuminator resonate with the vortex-energies of consciousness and by cross-acting generate the materialized thought-forms. The pineal-gland chakra seems to be where these vortex-energies 'synchronize'. Mental-enzymes generated in the nervous-system 'trigger' the production of the EE-units that the inner-self projects into the proper intensity-range for the thought-forms to manifest. Boyle's contribution to the emerging science of psionics is momentous, in the sense that we can project thought-forms into psi-intensity-masses via the so called occult-illuminator. To understand how this seems to be done, see the figure on the following page which represents a CHEOPS DOUBLE-PYRAMID in hyperspace, or what we could call a Cheops double-pyramid in psi-hyperspace.

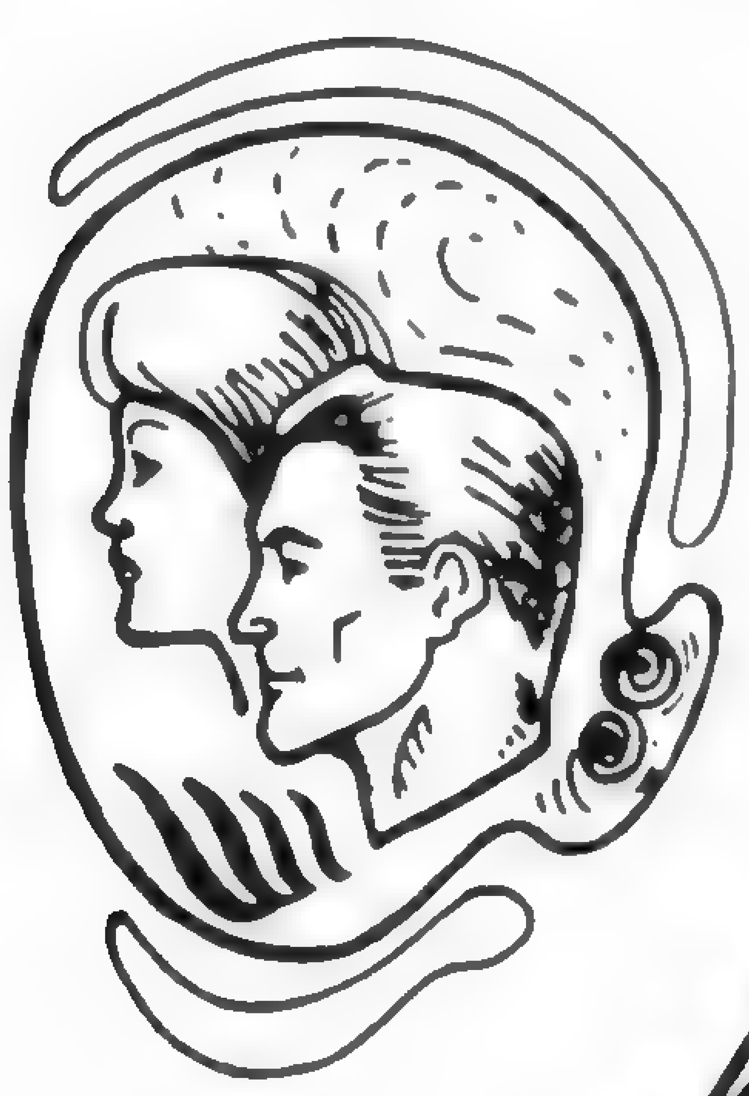
We take the hyperplane of the black-pyramid P-ABCD to represent a psi-phase-space of a given psi-intensity-mass. Now suppose we consider the vertex-edge PQ to be the path along which we move in psi-time, that is, for a very short 'pyschological-distance' via intuitive-associations using the inner-eye and inner-senses... . We use the OUTER-SENSES and OUTER-EYE in visualizing the reflected-images from the mirror-reflector at the base of the occult-illuminator--this corresponds to the black-pyramid P-ABCD in the figure, where we have assumed that the outer-eye has its physical-focus at the point P, and the mirror-reflector being at the base ABCD of the black-pyramid P-ABCD; it being understood that this is a geometric-representation and we have not used a truncated-pyramid, but the psi-principle is what we are after. In the interior of the occult-illuminator we will see the reflections of the parts on the mirror-reflector as we view the object-symbols with our outer-eye lying in our hyperplane, and in the figure, will be that of the hyperplane of the black-pyramid P-ABCD. Now if we go into a trance-like state by inner-focusing using our inner-eye which 'activates' the psi-time process, then as we move along the psi-time path we should be able to see with our inner-eye using our inner-senses materialized thought-form scenes from both the past/future as probable-events in different psi space-time frames of given psi-intensity-masses, and in some instances, the thought-forms will coalse/overlap, that is, the thought-forms may dissolve and form new psi-energy-gestalts of different thought-form patterns like in a kaleidoscope. If we go into a deep-trance, then the invisible-pyramids formed become visible via the innner-eye as we move along the psi-time path, and in the corresponding figure, the black-pyramid P-ABCD will be replaced by the red-pyramid Q-ABCD lying in the hyperplane determined by the point Q and base ABCD , being the other end-pyramid of the Cheops double-pyramid PQ-ABCD--it being understood that the red-pyramid Q-ABCD now represents to the inner-eye a psychic-image-pyramid lying in the hyperplane determined by Q and ABCD. We say, then, that the EE-units intensify and project the thought-forms into psi-space having a given psitronic-intensity-mass. Now at any point of the vertex-edge PQ, we shall have an apex-point of no-time which focuses the psi-energies which are being projected from the base of the Cheops double-pyramid in the probable King's x-chamber's, more so, from the mirror-reflector which has a focal-point at a point of the axis-plane OPQ. In fact, we have an axis-plane along which the inner-eye travels from the hyperplane of 1 pyramid to another. The bases of all these psychic-charged psitronic-fields forming the energy-gestalt-pyramids with the vertex-edge

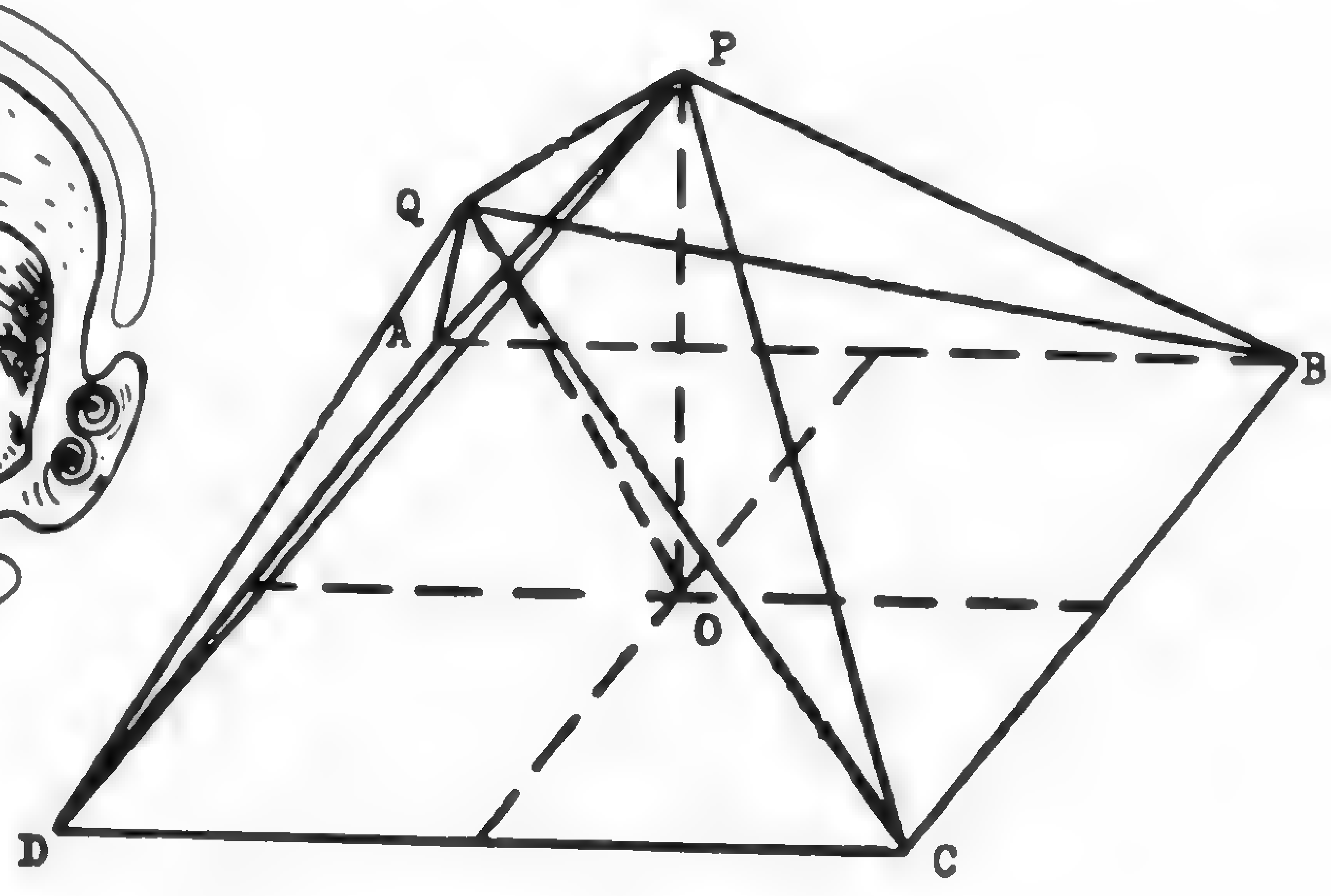
will have a base in common, which also lies in the hyperplane of the black-pyramid (like the rotation of a hyperplane around 1 of its planes), and all the apex-points will be at different psi-time points along the vertex-edge. We take the thought-forms materialized at, say, the other end-pyramid Q-ABCD, then the thought-forms will be in either the past or future at this x-level of psi-time focusing, that is, the degree of the trance-states. These results do show, indeed, that we do make use of the inner-senses using psi-time with the inner-eye as the psychic-focusing-apparatus.

The EE-units gathered into energy-gestalts of a certain intensity-mass will materialize as thought-forms along the psi-time track in psi-hyperspace. As the 2 vortex-energy-fields between the non-physical-mind and occult-illuminator via the brain resonate into different psi-frequency-bands, then psi-states of awareness will project different materialized thought-forms as such.

We could modify the pyramid-energies and ACTUALLY create a miniature-figurine in 'solido' in the King's chamber, something like a thought-form HOLOGRAM. Somehow that portion of the 'pysche' called the ID would be involved in creating the thought-form hologram where psi-id energies are involved and interact with the pyramid-energies; that is, 2 intertwined vortex-energy-fields. In fact, we need not actually use a pyramid-form of large-size, but a pyramid-energy-plate would work as well, but the pyramid-energies decay with 'time', whereas, the pyramid-energies of the geometric-forms last indefinitely. In fact, Flanagan has developed a pyramid-energy-plate that works quite well, although the decay-rate of the pyrsmid-energy-plate dissipates its pyramid-energies within 2 years or so, more or less. He also has developed a pyramid-energy matrix-grid generator that lasts indefinitely.

In these few pages of the addendum we have merely touched upon hyperspace-concepts related to the matter-spectrum. But one thing we do know for sure. The Cheops pyramid is the opening door to the inner-dimensions of realities. Much needs to be done along research-lines on pyramid-energy and its subenergy-fields of which we know very little of. We then consider how the vortex-atom relates to other intensity-mass fields. Anti-gravity becomes a practicality once we have considered the 'Venturi-effects' of interacting vortex-atoms, or what is the same thing, how different fields of vortex-pressures interact. It is indeed a strange universe when U consider that for thousands of years no one has bothered to consider the Great Pyramid of Egypt other than a historical-piece. But the latest data and theories prove otherwise. In a simple miniature Cheops pyramid we have the 'riddle of the universe'. The Sphinx would be saying to our inner-senses, "Look within for the secrets of the universe." I conclude this short-treatise by saying, "We have barely scratched the knowledge-levels lying latent within our inner-selves." So be it!





CHEOPS DOUBLE-PYRAMID

Scientists Make Amazing Discovery

- H/MATH Book CONTAINS....

The Miracle Power of the Fourth Dimension

by J. Allen Hewitt

Mystics and psychics have long believed in the power of the fourth dimension — the power to work miracles, travel through time, perform healings, attract wealth and success.

Now, research scientists at Brown University have made an amazing discovery that may bring respect to such "far out" ideas. Mathematical blueprints for the fourth dimension were fed into a special computer and, to the amazement of all, the computer revealed a dazzling figure of a 4-D Hypercube virtually identical in concept to an ancient occult symbol used by a 13th century Catalonian mystic and numerologist.

The swirling image of the Hypercube caused great excitement in the scientific community. A special presentation was made to the American Association for the Advancement of Science in Washington, D.C. and to the International Congress of Mathematicians in Helsinki, Finland. No one had ever "seen" the fourth dimension until this staggering breakthrough.

Mind Development, Inc., inspired by the work at Brown University, began specialized research into the mind expanding power of the Hypercube. The first step was to "clean up" the blurred, rotating image from the computer into a concise clarified version suitable for examining its mystical properties. The primary challenge was to retain the effect of multilevular motion within the Hypercube without the use of film, video tape or special effects. It took several months of experimentation, but the final result was an astonishing success. The secret was to use two overlapping images of the Hypercube color coded to the spectrum sensitivity of the rods and cones in the human eye.

Psychic Power Released

Did the symbol of the new clarified Hypercube itself hold the key to releasing the awesome power of the fourth dimension? That was the question that faced the research team at Mind Development, Inc. After consultation with clinical psychologists at the Human Awareness and Potential Institute, Mind Development, Inc. began a series of tests using galvanic skin response units and other indicators of altered brain wave activity. It was discovered that concentrating on the Hypercube, even for a few moments, induced altered states of consciousness — the same altered states of consciousness associated with psychic power, increased intuition, the ability to work miracles, even bend metal as demonstrated by Uri Geller.

Participants in Mind Development research have demonstrated their ability to perform psychic readings, mind projection, ESP and mental telepathy with astonishing accuracy while using the Hypercube as a focal point.

Free Hypercubes

"We have reached a point in our Hypercube research where we are very anxious for the general public, especially those predisposed to psychic potential, to conduct their own experiments with our special Hypercube," stated a Mind Development, Inc. spokesman. "In fact, we will provide a free Hypercube to everyone who desires to participate with us in this project as a member of our ad-hoc research team."

To become a member of the Mind Development, Inc. ad-hoc research team, receive background information on the activities of Mind Development, Inc., a periodic newsletter, plus a free hypercube identical to the one used in laboratory research, simply send your name and address with a $10.00 membership registration fee to:

Mind Development, Inc.
Suite 124
515 - 116th N.E.
Bellevue, Washington 98004

Note: A.G. Merklingar, founder and president of Mind Development, Inc., has been featured in Life Magazine, National Observer, Mademoiselle, Sports Illustrated, Denver Magazine, International Herald-Tribune, and on TV's David Frost Show.

377994. THE WORLD OF M. C. ESCHER. Text by M. C. Escher and J. L. Locher. 202 Illus., Many in Full Color. Insightful text and commentary accompany the most important of Escher's prints. Chronologically arranged, these drawings reveal the strange and highly individual world Escher created; a world where science and art merge, where reality is both wondrous and incomprehensible. 7¾ x 11.

From Brandes Letters Collection —

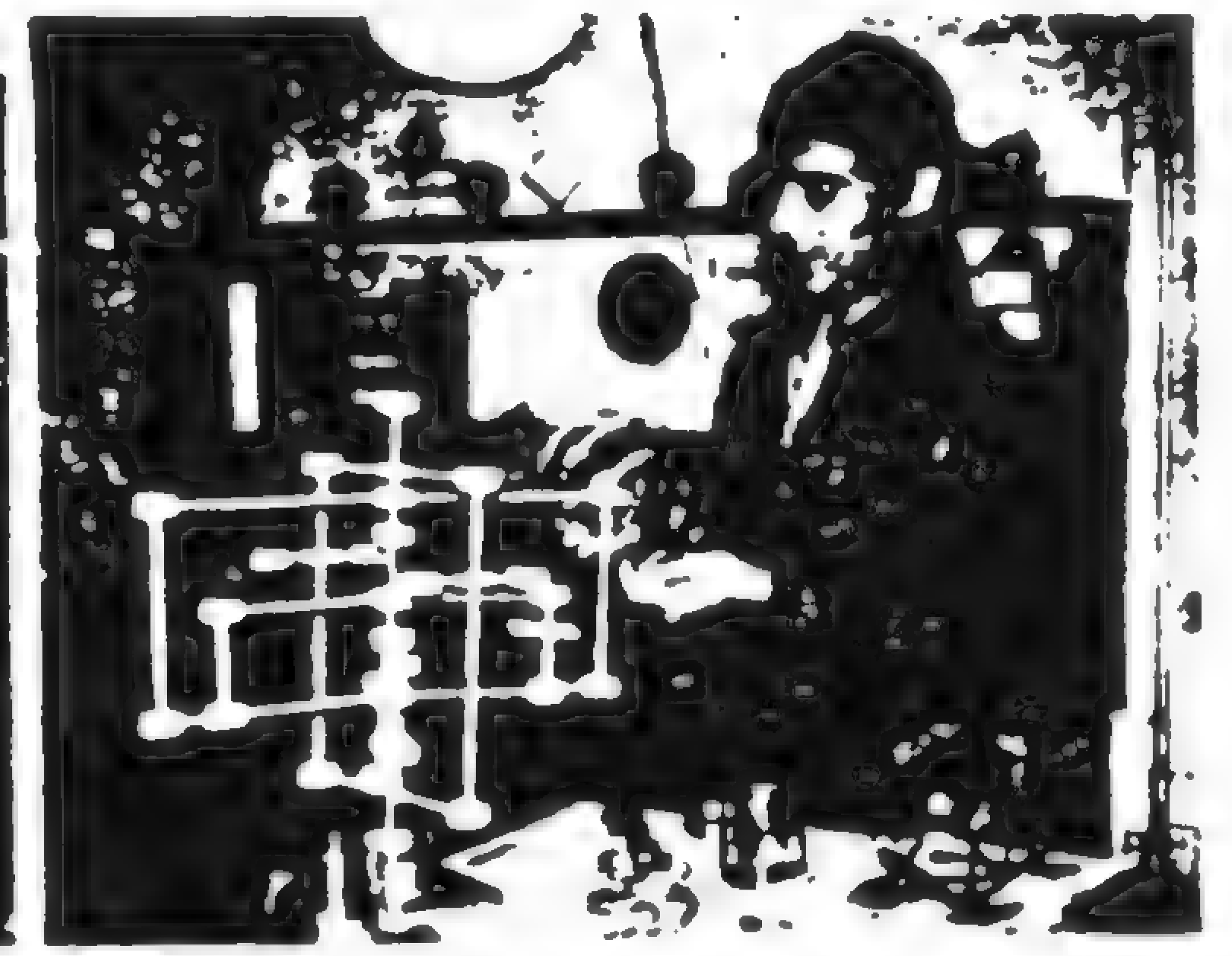
In the lower left foreground lies a piece of paper on which the edges of a cube are drawn. Two small circles mark the places where edges cross each other. Which edge comes at the front and which at the back? In a three-dimensional world simultaneous front and back is an impossibility and so cannot be illustrated. Yet it is quite possible to draw an object which displays a different reality when looked at from above and from below. The lad sitting on the bench has got just such a cube-like absurdity in his hands. He hazes thoughtfully at this incomprehensible object and seems oblivious to the fact that the belvedere behind him has been built in the same impossible style. On the floor of the lower platform, that is to say indoors, stands a ladder which two people are busy climbing. But as soon as they arrive a floor higher they are back in the open air and have to re-enter the building. Is it any wonder that nobody in this company can be bothered about the fate the prisoner in the dungeon who sticks his head through the bars and bemoans his fate?

THE GRAPHIC WORK OF M.C. ESCHER, by Ballantine Books, New York (East 50th Street New York 10022)—Escher's works as an artist parallel my own research in pure-math on hyperspace visual-grpahics, that is, I feel much in Escher works akin to my own creative-math research as such. Fascinating-prints...76 in total...

A SHORT COURSE IN TIME...

By Stan Tenen

By profession Stan Tenen has been an engineer, inventor and local talk-show celebrity, responsible for bringing the "Prisoner Rap Sessions" to KQED. By nature, Stan is an eclectic thinker, a first rate intellect with the ability to pull together and correlate insights from history, physics, geometry, mythology and mystical tradition. He has uncovered a collection of complex but easily comprehendible patterns in ancient teachings and symbols that point to a common origin for all religions and decode fresh universal insights. Stan's talent for explanation borders on the amazing, actually immersing his listeners in the experience of creation and regeneration, leading one to the precipice of another dimension, into time and endless possibilities. There are truths here, not to be accepted as articles of faith or prophet's edict, but on the power of the understandings they'll open up in your own mind. Those of you who were ever inspired by Peter Pan, Don Juan or Albert Einstein ought to check this out. (See Stan's course in the Potpourri section on page two.)



Once upon a time about 12 years ago I happened to open a copy of Genesis I had had for many years. It was in Hebrew. Now, I don't read Hebrew although I can recognize the letters of the alphabet. If I could have read the Hebrew, none of this would have happened. Who looks for or notices a "pattern" in the letters of a text they can read? In my ignorance, instead of reading the Hebrew words, which I couldn't, I thought I saw a "pattern" to the letters.

I was startled.

As far as I knew what I saw was impossible! The implications were staggering. Here was the root document of Western Civilization, the basis of the Hebrew, Christian and Islamic faiths, sacred even in "translation," and I just noticed it contained, somehow, a hidden mathematical pattern.

At that time I knew nothing of the origin, history, or mystical tradition(s) of the text. After failing to "pull out" the pattern quickly, I realized I had some homework to do, and I started reading and asking questions. The more I read the more the mystery deepened. I learned about the many mystical legends about the text, something of the geometry and mythology of the Great Pyramid at Giza, a little of the Vedic/Sanskrit tradition(s), etc., and the Hebrew mystical tradition called Qabala which claims the Genesis text is coded. Try as I might, though, I could not explain this pattern on the basis of the Qabala nor any other of the traditions I researched. The pattern I could see was fundamentally different. Three years ago, with the help of the KQED "Prisoner Rap Session" audience, the code cracked. This course is what spilled out.

This is indeed an entirely new and very powerful perspective. If any tradition explaining this material has survived and been published in English, it has not been made available to the uninitiated.

The implications are truly staggering even from the parts decoded so far. The elements of Egyptian, Greek, and Sanskirt cosmology merge naturally with the mystical traditions about the Great Pyramid, and the religious traditions of the ancient world. These ancient mythologies and cosmologies can be identified uniquely and directly with what modern science now tells us about quantum physics, genetics/DNA, and the Einsteinian saddle-shaped universe. (Note to the scientifically timid: All of this is geometric and I have built lots of "tinker-toy" models. You can understand the "code" with only the math now taught in Junior High School. The models are recognizable.)

This teaching differs not only in content but also in form from traditional religious interpretations of the "sacred." It demonstrates an entirely new theory of the origin of the Hebrew alphabet and provides the missing links between the traditional schools. It is not based on personal revelation, automatic writing, or a trip in a flying saucer. (Such personal experiences are not normally shareable.) It is self-decoding and self-teaching. The decoding is based on the same logic as was used by the non-mystical NASA-Jet Propulsion Laboratory scientists to encode the Voyager and Explorer spacecraft plaques and the Arecibo, Puerto Rico radio telescope-message to outer space. These attempt to send a greeting from Earth in the universal language of math.

Perhaps most startling of all, this rational, logical decoding verifies the religious and mystical claims made for and by Genesis by believers. It provides, among much else, a scientific, rational basis for out-of-the-body experiences and time-travel.

Understanding this material evokes the unique personal experience known as Initiation.

Be seeing you.

A SHORT COURSE IN TIME... $30
Stan Tenen 456-0487/673-0023 X97

A: Laguna & Marina, Room 205, S.F.—3 meetings Tuesdays 8-11 p.m. starts January 6.

B: 1741 Virginia, Berkeley—3 meetings Sundays 12n.-3 p.m. starts January 11.

...Or an irreverent demonstration of the Ancient Riddle of how to Square-the-Circle in twenty-seven easy steps—you may count them yourself—without really dying. Discussion will center on an entirely new, unpublished, decipherment of GENESIS in relation to the Great Pyramid, Qabala, the genetic code, the Sri Yantra and modern physics. Call for information. Section A: at Fort Mason.

Stan has been a consulting engineer/physicist. He conducted the "Prisoner Rap Sessions" on KQED-TV in 1978 and is now writing a book on the coude material.

tion.

THIS IS A LETTER FROM 'H' MATH DISCOVERER..MORE LETTERS ARE AVAILABLE...BOOK IS ONLY FOR PERSONS WITH BACKGROUND....

MATHEMATIQUE-supreme

12-12-77

BRANDES LETTERS COLLECTION (HUGE & ASTONISHING) $15.00 FRY'S

Can't contain my enthusiasm for the rest of this year... I give U 1 more-supreme-discovery on the ternary-typo-prime-arithmetic (TTPA). In music, fugues, counterpoint, harmony correspond 1-1 'topologically' in a sense to residues, compositions, syzygy in the prime-arithmetic respectively. This I discovered tonite, indeed, the prime-arithmetic of prime-resolute-numbers the symphony of arithmetic. Combinatory Analysis in some of its x-aspects forms the counterpoint for the prime-gestalt-structures; the residues the fugue-themes, and the harmonics as syzygies—beautifully 100 +, period.

Cometary-matter as in the Secret Doctrine by Blavatsky verified in the prime-arithmetic organizer-theory 100%. Cometary-matter into embryonic-stars shooting forth embryonic-planetary-matter, then the embryonic-cometary-matter forms into a blue-white star with its satellites of revolving planets—live-stars and live-planets.

But the prime-arithmetic goes much deeper in cosmogenesis by far, for the prime-arithemtic postulates that there exist not 1 atomic-scale of the atomic-elements, but an un-limited-number of atomic-scales within a matter-spectrum, and U end up with different frameworks of unique, distinct atomic-elements in other star-systems, some similar, others dis-similar—the Fohatic-organizing-force becomes x-Fohats in other x-frameworks. Our atomic-elements as organized in our 3-space cell frame merely local-atomic-matter-groupings, period.

ATOMIC-BARRIER

In a star-ship traveling at warp-speed in 'hyperspace' (elliptic-hyperspace of Riemann), then the star-ship 'crew' must translate their physical-image-bodies on entering another star-system with a different atomic-matter-grouping, the re-programmed physical-image-body would then adapt itself atomic-wise in the new-matter-grouping; otherwise, a non-translated image-body would dis-integrate upon entering another star-system of different atomic-groupings of electromagnetic-energy-fields from pre-matter EE-units into its intensity-mass 'x-matter'... The electromagnetic-spectrum is enormous, we have only discovered but a paltry-few x-aspects of inner-electromagnetic-energy...

The prime-arithmetic implies that there exist an infinite-number of local-matter-groupings of manifested atomic-elements in x-frameworks. Our Western-Science is DEAD NOW 100%+. So be it!

In 1 of my 'dream-projections, my inner-eye saw 'entities' having electromagnetic-image-bodies of awesome-potentials, a glimpse of their 'reality' made-me acutely-aware that multidimensional-consciousness-units create un-limited numbers of x-bodies to manipulate with. In any x-realm, 'consciousness' mush be enclosed within a container (or capsule) to keep its 'energies' from seeping away. So literally then, in other x-frameworks, x-consciousness-units will have corresponding x-bodies in which to manipulat their x-environments—Out-Of-Body projections shows this to be true 100%...

Priestcraft is dead on this planet! The antrepomorphic-gods created by our Earth-species is truely a joker-in-the-deck of illusions, and becomes obvious in Out-Of-Body projections to the inner-planes. One could spend a 'lifetime' just studying 1 of the infinity of inner-planes, period.

The 5th-root-race is finished, being only an outer-reality-manifestation-type of aware-consciousnesses. The coming 6th-root-race now manifesting in embryonic-form will be inner-outer-manifestation-types, and the pause$_2$ of pause$_1$ of 'reflection' (doubly-circuited synapse-cel. The double-triangle

corresponds in hyperspace to a DOUBLE-PYRAMID as in the H-book. (in occult-texts black-white-triangles to black-red-pyramids)

Sincerely Yours,

George L. Brandes

I began the hyperspace-math research in 1955, then some vague 'hints' of what was to come, in the early '60's. Then spent 15 years on-and-off developing the visual-hyperspace-math. In fact, 1 close friend of mine helped via my'feedbacks' to clarify many hyperspace-concepts. I spent hours and hours trying to explain to my close friend 'Wendel' that 'we' who have bodies of clay AND A SPIRIT CAN VISUALIZE THE 4th-DIMENSION 100%. How? My friend asked. My argument was, 'We can, with just 1 'clue', use the INNER-EYE AND INNER-SENSES THEN MAKE OUTER-INNER-EYE GRAPHICS and said results should be 100% effective. Well! I was right using this kind of 'inference'. After a few years 'teaching' my friend Wendel the 'mysterious-world of the 4th-dimension, he finally make the 'quantum-jump' using his inner-eye and was able for the 1st-time to 'visualize' 4-space perceptics, though with some 'difficulty' at the 'time'. Now, I knew for sure that I was on the right-path, for if I could teach just 1 highly-intelligent-individual some visual-hyperspace-math, then not why a whole planet? I said yes, if and only if I could put it all down in writing, a gargantuan-tast alone to say the least. So after 8 torturing months of 10-15 hours per day, I finally finished the H-manuscript--though a lot is still to be desired. I could write a large-volume alone on the H-math, but cost-of-printing would become prohibitive, and so settled for 'basics' only, that is, 132-pages of H-material with

INSERT

NEVER, but a hyperspace-engineer, yes." But knowing my wild-talents, I settled on being a hyperspace-mathematician instead.

The saying, "KNOW THYSELF", is true today as it ever was, meaning, ETERNITY for any 'evolving' SOUL or ENTITY. My'God of clay is dead', the new nameless one that I take for my new-belief is called ALL THAT IS. So be it!

I aim for the stars now. Perhaps soon, Von Braun's dream may come true. I still have his old-letter in a safe-deposit box.

Now I will open your inner-eyes some. Just for analogy only imagine the following:

1 3-space cell called Earth .

more so (flatland-analogy)

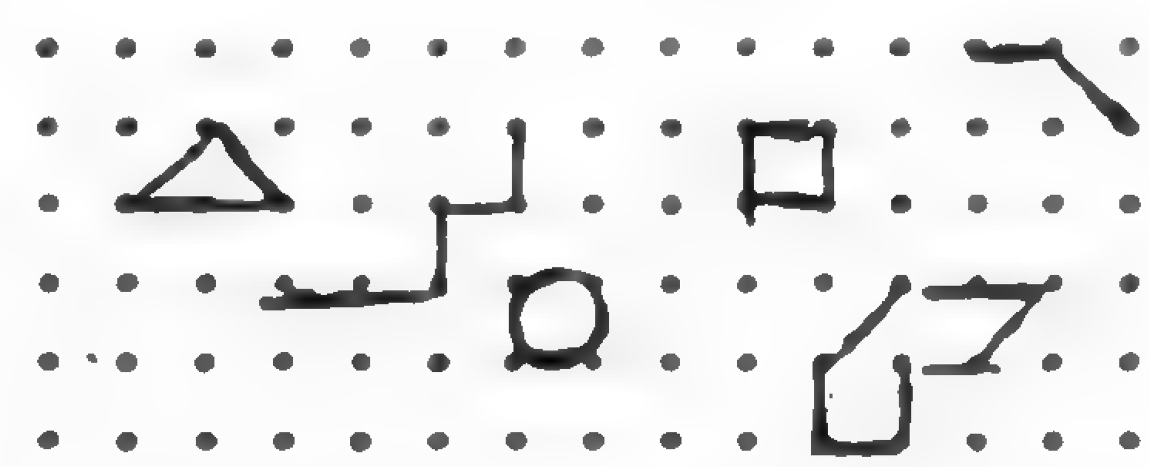
G-gestalt-patterns unrealized due to 3-space viewpoints....

2-valued logic, closed-systems, dualities, etc.

An Earth God of Clay, physical-image takes the 'form' of what the PROJECTIONIST THINKS, a 'God of clay'...how absurd?????

In a 4-space level-gestaltung, U are OUTSIDE LOOKING IN... EXPANSIVE-INSIGHTS ARISE OUT OF NEW INNER-DIMENSIONS. The mental-pace becomes enormously ACCELERATED— psi-time encompasses many x-realities such as probable-past, probable-futures

n 3-space cells of a galaxy(galaxi



gestalt-patterns realized due to 4-space viewpoints... n-space viewpoints...

open-systems, n-valued logics, gradient-scales used, etc.

A Universe of ALL THAT IS... UNSPEAKABLE... What U see, U project into x-realities— power of the multidimensional-personality called 'soul' or entity.... draws its 'energy' from ALL THAT IS.

P.S. The above hyperspace-analogy is very crude, the emphasis, to stress a comparison between a 3-space viewpoint and a 4-space... viewpoint (viewpoints as such).

Sincerely Yours,
George L. Brandes

P.S. When U get a free-copy of the H-book. I must warn U ahead of 'time'

. . . To psionicist, the H-book will be a 'life-saver' for many of their psi-viewpoints. The H-book will bring the end to the 'order' of the materialist-scientist once and for all, by this I mean, we 'exist' in an infinite-matter-spectrum of an infinite-number of DIMENSIONS, or simply, x-realities, not 1, but polyrealities of many dimensions. those who seek 'quite-answers'. An invisible-door is now OPENED GEOMETRICALLY TO THE 4th-DIMENSION. Good Luck! 1001 hyperplanes of a simplo hypercube, ha!

Resistor with a Real Twist

HOW CAN you squeeze troublesome inductance and reactance (resistance to changes in AC) out of a resistor? One way is to make a resistor in the shape of a Moebius loop—a century-old mathematical oddity that is based on a geometric surface having only one side and one edge.

Under ideal circumstances, a resistor should provide only resistance, a capacitor only capacitance and an inductor only inductance. Unfortunately, in high-frequency circuits (UHF and microwave) and especially in pulse applications such as radar, the design and operation of such circuits is greatly affected by the unwanted reactance inherent in these components. The higher the frequency, the more critical these parasitic values are.

A unique solution to one of these problems (making low-value resistors non-reactive) has been found by Richard L. Davis, an electronics engineer with the Sandia Laboratories in Albuquerque, N.M. Davis reasoned that if current passing through a resistor could be divided into two equal components whose electromagnetic fields cancel out, the reactance of the resistor would be small. How could such a resistor be made? The Davis solution was to add a simple Moebius twist to a ribbon- or wire-conductor resistor.

Kooky Loops. Perhaps the best way to visualize the construction (and operation) of a Moebius resistor is to make a couple of Moebius loops from long strips of paper that are about an inch wide. First make the basic loop by joining (with tape) the two ends of a single strip after you have given the strip a half twist. This loop has only one surface! Prove this by drawing a line along the full length of the strip, right back to your starting point (see lead photo). The line will cover both sides of the strip.

A Moebius resistor, however, must be constructed with two conductive ribbons, with or without a separating dielectric layer. So now make another Moebius loop, this time using two identical strips of paper, one on top of the other; again, give the strips a half twist before joining the opposing ends together. Label one of the splices *input*, the other *output*.

It may appear that there are still two separate loops—a pencil between the strips can be slid completely around the loop back to the starting point. Actually, there is just one loop. You'll see this when you attempt to separate them. The two paths that the input current will take to the output terminal can be traced once the loop is opened.

How It Works. The input pulse that's applied to one of the terminals divides into two equal components which travel in opposite directions. This happens because the impedances of the two paths to the output terminals are identical. Since one pulse loops to the right while the other heads left, they cannot interfere with each other. Then, when the pulses have traveled half way to the output—where DC resistance is one half the total value—the pulses are at equal potential and of opposite phase. By the time they reach the output, their potentials fall to zero.

The two terminals must be exactly opposite each other otherwise the resistor becomes inductive (the pulses wouldn't be 180° out of phase and residual magnetism would be present). While it is preferable to eliminate lead wires whenever possible (to avoid stray capacitance), a resistor that is slightly capacitive can be nulled into balance if you adjust the lengths of the leads.

Davis' first experimental resistor was made of aluminum-tape conductor placed on masking tape. The masking tape serves as the dielectric. It had a 0.022-ohm resistance and 0.003-μh residual reactance. The time constant (1.3×10^{-7}) was very low for such a small resistance. These values may seem ridiculously low to people who experiment at audio and lower RF frequencies, but as you get up into the spectrum such component values have tremendous effects on a circuit.

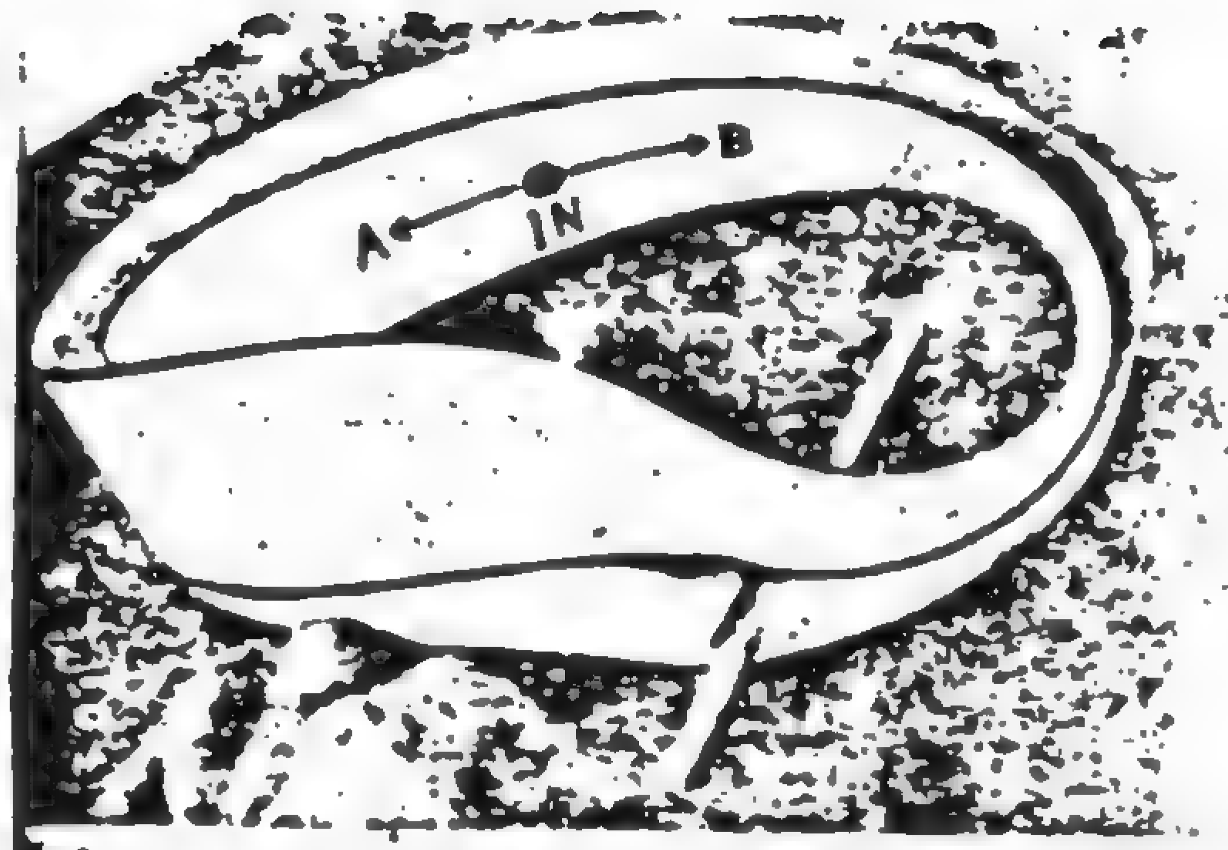
Tuned UHF circuits have resonant frequencies requiring almost invisible capacitors and coils, and the short wavelengths are too large for most any component. In fact, most radar circuits use resonant cavities rather than individual capacitors and coils. Cavities and waveguides act on electromagnetic fields, while resistors, capacitors and coils are designed primarily to control electrons flowing in wires. The former act like distributed constants, the latter are lumped constants. A Moebius resistor is a lumped-constant component.

Great Versatility. One striking feature of a Moebius resistor is that it does not couple electromagnetically to other metallic objects or to itself, even if the shape of the finished resistor is changed. There are only two requirements for this: the conductors must not touch physically and the spacing between the conducting layer must not be altered. This non-coupling characteristic makes it possible to wrap Moebius resistors around cards.

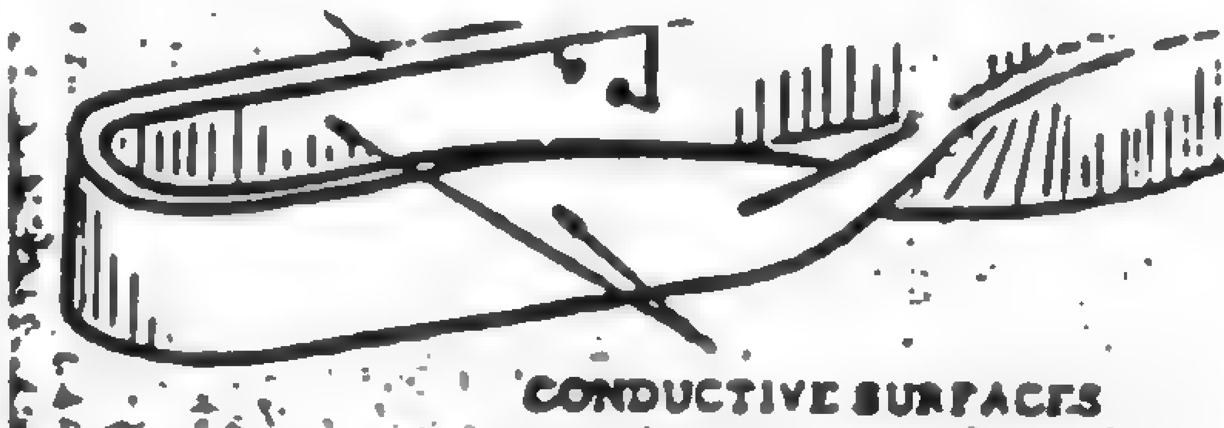
Moebius resistors are simple—and inexpensive to make. Problem is, unless you've got a rig that works at frequencies from around 500 to 4000 mc, you won't find much use for them. Of course, if you're a mathematician you can always reach for a textbook on topology—just to find out what Mr. Moebius was really talking about.

By JORMA HYYPIA

Nov - 69
ELECTRONICS ILLUSTRATED
Fawcett Bldg., Greenwich, Conn 06830



Two paper loops simulate Moebius resistor made of two conducting ribbons and separating dielectric. Input and output terminals must be opposite.



Moebius loop resistor has only one continuous conducting surface. Dielectric material separates opposing surfaces of the conductor. Leads are exactly opposite.

Richard L. Davis, inventor of Moebius resistor, shows how this component looks before it's bundled into a compact package.

Actual prototype of Moebius resistor is shown at right. The resistor strip can be wound into a compact bundle without adversely affecting its performance. This item may be next design.

Cut in 110-1 R

The most-imortant letter to date
on starship-propulsion-systems
It will UNGLUE the entire
Scientific-Community of Flatlanders.

3-30-83

Dear Fry:

Got it now! Hyperspace-travel via magnetic-force-fields, tis 'all' clear-now 100%. What kicked it off was a letter from Glenn J. Beaumon The III. I was trying to figure out this guy's address from his long-hand writing...etc. To reply I need his address, but he lives in Denton, . I figured from his cipher-scrip using 3 of those 'square-root x like signs, 3 alike, to mean he's from Texas—to be sure I will check it out at the post office Monday morning.

Now then what keyed-me-in was what Glenn said in his letter about possible propulsion-systems—I jumped to the inner-intuitive-level to evaluate his emotional-feelings on the matter, found he was still in the dark by far, far. In fact nearly 'all' researchers today have missed the KEY-factor in hyperspace-propulsion systems. Yes, magnetic-force-fields triggered-off via fusion-generators, but there exist one extremely-important-clue that I discovered today on how to begin designing starships for multiple-light-velocities beyond c into ranges from

$$c^2, c^4, \ldots, c^{512}.$$

It is so utterly simple in concept that I wonder why I had not seen it before. Here it is as follows:

> U change, literally, the space-time frame of reference about the spacecraft to another space-time frame of reference which implies a corresponding change in dimension of the space-time coordinates about the spacecraft. To bring about an alteration of dimension(s) about the spacecraft it must be encapsulated with a magnetic-field within the 5 million Gauss range of field-intensity (flux-field), then with the cancellation of INERTIA about the spacecraft the next step requires warping the magnetic flux-lines of force into parallel for an instant, and each surge-jump of the magnetic-flux-field propells the magnetic-spacecraft into the next higher-dimension via a POWER shange of the Planck-constant. The serial-universe dimensions interlock in quasi-matter-states of proportionality via the Planck-consta[nt] power multiples. When U change the magnetic-force-field intensities U literally alter the frame of reference of the magnetic-field(F_1) into a new interdimensional frame of reference(F_2), etc.

Taking the different frames of reference we have this 1-1 correspondence with multiple-light-velocities as follows:

$$\begin{matrix} F_0, & F_1, & F_2, & \ldots, & F_9 \\ c\ , & c^2, & c^4, & \ldots, & c^{512} \end{matrix} \quad : \text{this 'relatively'so} \ldots$$

I need not say more, however, to bring about changing of the Planck-Constant to higher-power multiples(dimensional-timing-principle) into a physical-translation of theory into practice it becomes imperative to have FUSION-GENERATORS to build-up the very-intense MAGNETIC-FORCE-FIELDS required for hyperspace starship travel: no fusion-generator, no starship propulsion-system, no alteration of frames of reference interdimensionally of the space-time coordinates: magnetics, rather, 9-magnetics the key in altering the frame of reference of space-time coordinates(the pea-brained Einstein never saw with his 'old-eyes' of 'perceiving' other x-realities: in fact, the H-book tells us that on the 4-space level, the 3-space frames of reference are infinite, etc).

New-astounding breakthroughs on the prime-arithmetic—have now around 312-pages and still going strong with it, maybe someday I will publish it in compendium-form after abstracting-out the core-material around which it is built.

Sincerely Yours,

George L. Brandes

R E V I E W S

Bridges to Infinity

ARE YOU ONE OF THE MANY people who feel absolutely helpless when faced with higher mathematics? "Math anxiety"—as it is called—is often prevalent among non-technicians. Even sophisticated, intelligent persons, who are otherwise capable and self-confident, break down in confusion when faced with mathematical formulas.

Michael Guillen teaches a course at Cornell University called "Math for Poets." His new book, *Bridges to Infinity—The Human Side of Mathematics*, opens up the world of mathematical concepts to the lay person. It will not teach you the symbols and mechanics of math, but will allow you to understand the fundamental ideas, and "see" the concepts that shaped math history. He gives this history in a series of short essays—each dealing with a different concept.

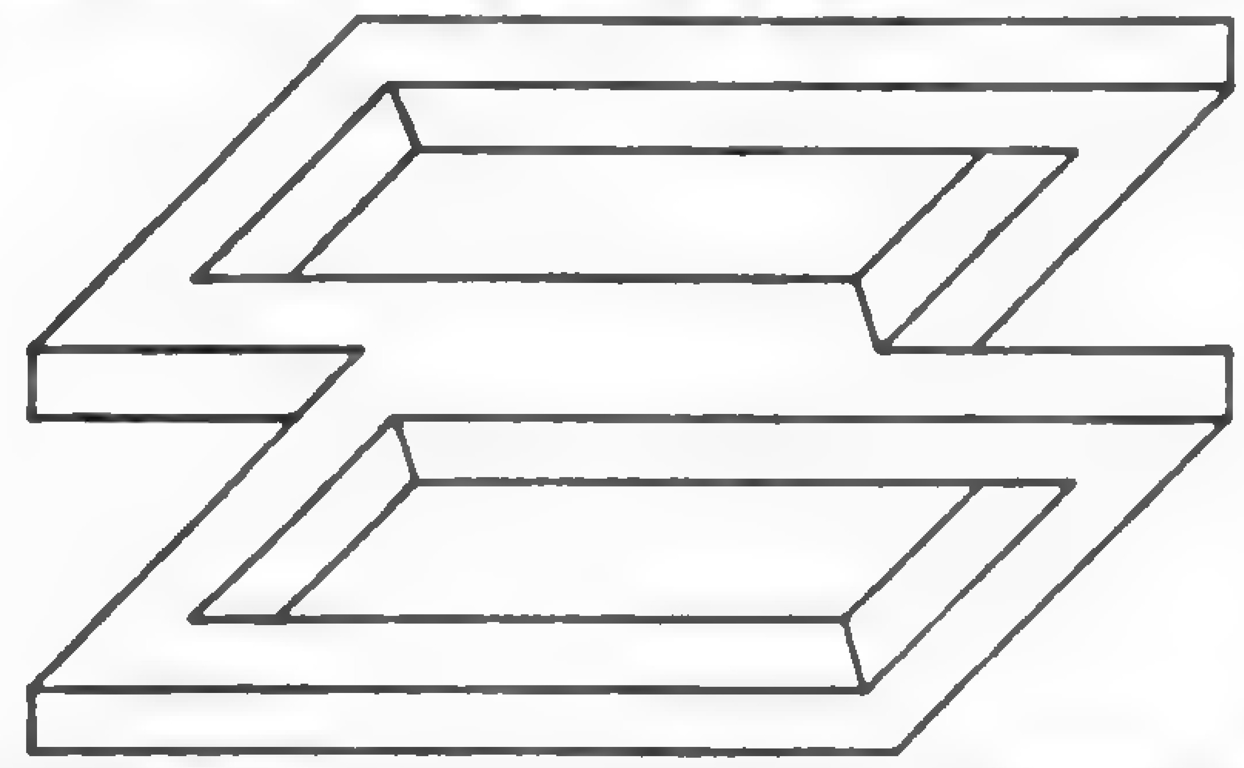
Guillen labels the early history as the age of "fantasizing." During this time, math was believed to be a perfect model of reality; although humans might err in calculations, in principle, math was considered infallible. Euclid had set the stage by providing proofs for all geometric properties, and mathematicians were working hard to show that arithmetic and algebra were on equally-solid ground.

> "There are other possible universes totally alien to our everyday experiences."

Between the mid-nineteenth and early twentieth century, two events shook the foundation of the mathematical world. First came the discovery that there are non-Euclidean geometries. Euclid's axioms and proofs had previously successfully described our "common sense" experience on earth. But, three mathematicians, separately and at about the same time, discovered that other geometries are logically possible. There are other possible universes totally alien to our everyday experiences.

According to Guillen, the impact of these discoveries on math was the realization "that mathematics is a mere invention of the human imagination and not a body of universal truths based on common sense, as everyone had believed."

The second blow was delivered to the old world of mathematics by the Viennese logician Kurt Godel in 1931. He showed that there are certain "unverifiable truths" that logic, by its very nature, cannot prove. Testing the very limits of logic, Godel surmised that there must be an infinite number of mathematical hypotheses that are extralogically true, but that defy being proved by logic.



The final section of the book deals with the modern age of mathematics, when mathematicians are now well aware of the limitations of their field, and are trying to "optimize" or make the best of their situation. This is one of the most interesting parts of the book because of the references to topology, catastrophe theory, game theory, and other concepts that are in popular usage, but not generally understood. The imagery evoked with the explanation of topology is particularly exciting. In each instance, the author goes to the one or two critical aspects of each theory and makes it comprehensible.

In conclusion, the book is excellent, but not without its faults. The essays—perhaps originally lectures—are sometimes too short and simple, giving only the most general notion of an idea. There is some repetition, although the book is written so that you could go to any single essay and learn about that concept without reading the previous sections. It also shows mathematics as a history of ideas, and how those ideas have shaped our world.

Dear Fry:

Here it is, suppose(DO IT PHYSICALLY!) U take a rubber-band and consider it as 1 MAGNETIC-LOOP of force, now stretch it out so that the rubber-loop is almost like 2, parallel-lines, let it go!, it will snap-back its 'loop' but in the process the rubber-band will sail like hell through the air in the meantime. It is ALL-IMPORTANT that U understand that energy was needed to stretch the rubber-band-loop. However, the rubber-band-analogy terminates too soon...

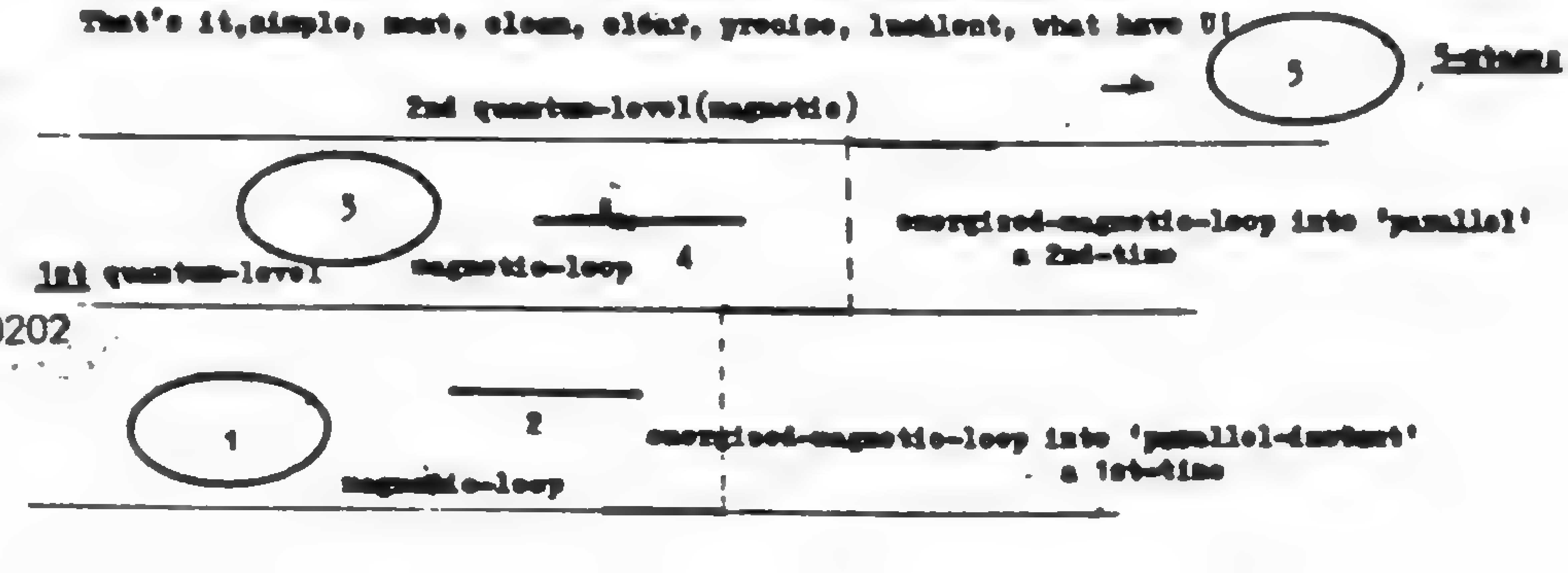
Now the biggie! Replace the rubber-band-loop for a magnetic-loop of force and the physical-energy supplied to the rubber-band-loop by a FUSION-GENERATOR. Boy! Is this thing BIG! Now as before use the magnetic-force-field created by the FUSION-ENERGY(self-sustaining free-energy) to 'bend' or 'snap' like a magnetic-loop into PARALLEL for an instant, then the jet-surge of magnetic-force is sufficient to take the magnetic-field (INERTIALESS) spacecraft to the velocity of light-squared to the 1st magnetic-quantum-level.

Now the MOST-INTERESTING ASPECT YET... Once the 1st magnetic-quantum-level is reached, the fusion-generator can supply additional-energy for the next-surge of magnetic-force by taking the magnetic-loops a 2nd time and snap them into parallel for an instant, resulting in a 2nd-surge of magnetic-force propelling our magnetic-field spacecraft to the 2nd-level squaring of c^2, i.e. $(c^2)^2 = c^4$; in actuality, the squaring will be somewhat more than c^4, more-like c^{4+}.

the Einstein-method of physics must be DROPPED, like shattered-glass!

Perhaps we could call it a vibrating harmonic magnetic-loop!

That's it, simple, neat, clean, clear, precise, lucid, what have U!



APPLICATION AND EXAMPLE SHEET FROM G.L.BRANDES LETTERS TO A. H. Fry

(The Brandes Letters Collection -$15- contains further examples)

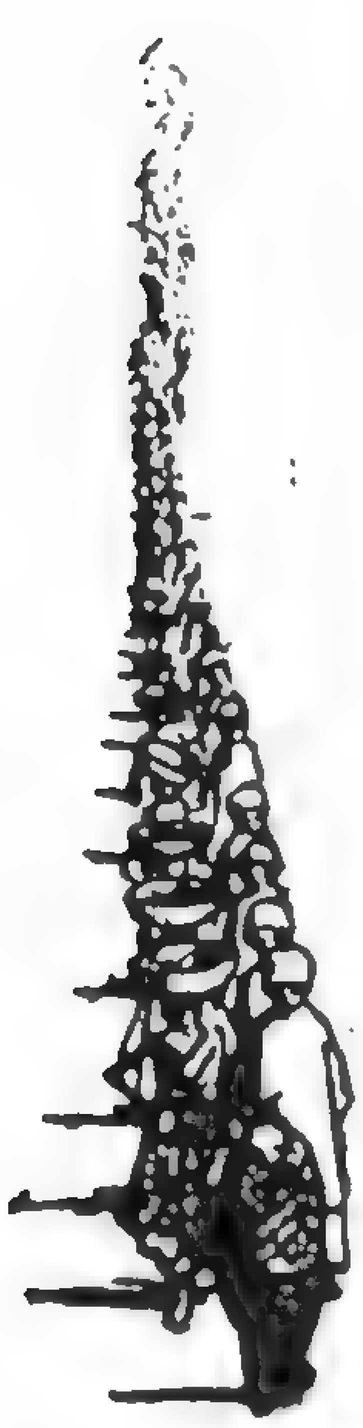
TIME as-we-know-it is an ILLUSION of the outer-senses. U change the PERCEPTION between events, and TIME will change also. A serial-sequence of events induces a linear-time-perception, acceleration changes the time-frame; but NOT ALL PERCEPTIONS used to classify event-happenings have a TIME-element, for example, OUT-OF-BODY PROJECTIONS exist beyond TIME and SPACE concepts as-we-know-of-it.

In my own OUT-OF-THE-PHYSICAL-BODY PROJECTIONS, I was ASTOUNDED to discover that TIME is an ILLUSION or camouflage within out 3-space cell—a root-assumption in this Pinda-Universe VALID only within a 3-space cell.

In Out-Of-Body PROJECTIONS via the PINEAL-EYE U can ESCAPE from this 3-space cell and visit the other DIMENSIONS within [illegible] and beyond. It has taken me many months to learn the para-mechanics of projecting my LIGHT-BODY out-of the physi body and enter the astral-borderland, soon now via the awake direct-projections usin the ALPHA-STATE trance-method, I will be able to enter the inner-planes by stepping 'inner-vibrations' higher and VANISH from this 3-space cell 'momentarily' to explore the wonders of the INVISIBLE-DIMENSIONS beyond TIME and SPACE as-we-know-it.
Our entire Scientific-structure COLLAPSES 100% once U have done a projection and left your physical-body to explore other DIMENSIONAL-REALITIES per se—it's for REAL
In the light-body U can FLOAT through SOLID-WALLS with the greatest of ease, and most astounding of all, travel in a blur of speed FASTER THEN THE SPEED OF LIGHT!

Not only that, but with still deeper-projections shed the light-body for one still LIGHTER in density, i.e. a causal-body, then a mental-body, further still to a SPIRI body devoid of the outer-sheaths—somewhat like a RIEMANN-HYPERSURFACE with the x-bodies wound around the physical-body, but unwind via 'silver-cords' in multiple-projection-states to inner-DIMENSIONS—in the light-body U can SEE with the inner-ey the vortex-like motions and pulsations of the physical-body, see through SOLIDS partially as transparent, etc., etc.

Most my 'free-time' now is spent on OUT-OF-BODY PROJECTIONS to other DIMENSIONS for higher-knowledges. I have 1 other super-ace in the hole, however. Ha! With the Out-Of-Body projections, BIG-BROTHERISM VANISHES completely as a power-structure, U go beyond EARTH-power-structures of the 'defunct' banker-politicos and their stupid games.



A few nites ago I pondered some implications on some probability-aspects of a surephere in 5-space frameworks which nearly blew my mind in raw-panic. I considered spherical-hypercyclinder of double-revolution in a surephere called a DOUBLE-HYPERCYCLINDER which contains an infinite-number of Clifford-surfaces(vortex-rings)—networks of systems of parallel great-circles, so astounding that I wondered the havoc that a system of vortex-rings an projected 'cones' do to the outmoded time-concept. The Einsteian-physics collapses in 5-space geometry frameworks, period.

The only possible way that I could resolve-out geometrical-properties of a surephere was to follow SETH'S extended 'time-concepts', i.e. NOT TIME per se, but CU-units having their own energy-pulse electrical-intensity-fields allowing for the probability for time-effects to thrush inward-outaward, backward-and-foreward into ALL probabilities simultaneously, and therefore ALL 'time-events' are SIMULTANEOUS. This gives rise to numerous new kinds of multidimensional light-phenomena 'effects', a continuum of light-fields, and this designation does not seem justifiable either...

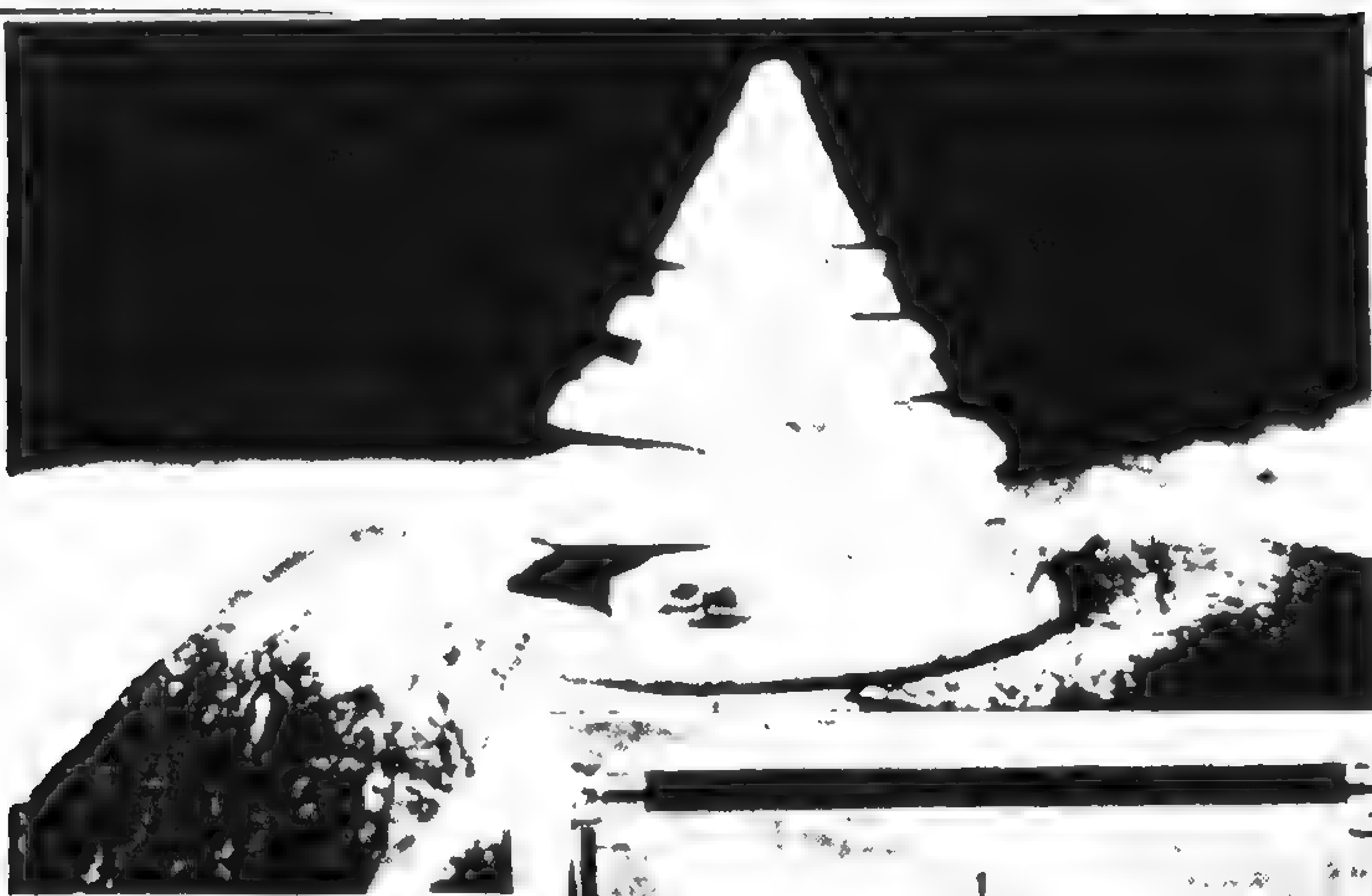
The vortex-atom in its multidimensional-aspects completely devastates the Einsteinian-physics. Tonite I completely part ways with the severely limited-concepts of Realtivistic-physics, in fact, some out-of-body projections was the 'catalyst' that shattered once and for all my 'quasi-beliefs' in the Einsteinian-physics. Many orthodox-scientist won't believe a 'word' what I say, but I will 'gamble' on my own INTUITIONS, OUT-OF-BODY EXPERIENCES , and some geometrical-aspects of 5-space geometry of the SURPHERE, etc.

So I will be doing some time-travel in the out-of-body state, and so for some strange impulse, I seem to be drawn to some 'alien-like' inner-vortex-plane where different laws operate in that 'universe', whatever it may be... Enormous changes in my personality-makeup has been happening to me lately in a very subtle-way as a result of the Out-of-Body projections, what I call a change in neuro-organization of brain-functions arising from another-level of 'reality', that is, the inner-stimuli induced upon the 'brain' via out-of-body projections has activated new-portions of the brain-functions.

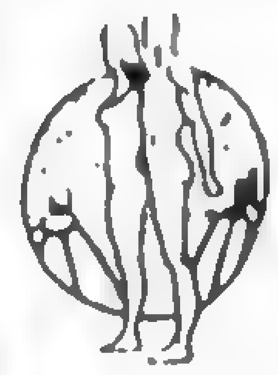
When CU-units enter a black-hole, 'time-effects' disintegrate, and when the black-hole turns inside-out into a white-hole again, new CU-units enter again building-up time-effect processes—the observed-effects of TIME and SPACE are very PLASTIC ...
ALL observable-events are the result of CU-units and their energy-gestalt-organizations, in other words, consciousness lies behind ALL manifested-actions, something that orthodox-sciences in its dogma refuses to accept as the basis for all x-realities.
The Earth is a living-body of consciousness-units also, believe it or not...!

Here is the great-secret(esoteric) to vast-discoveries in out-of-body projections: ALL inner-plane 'actions' are the byproduct of CU-units. ALL THAT IS manifest itself as probable-gous and continually re-creates new-universes to explore new-creativities of ITS MULTIDIMENSIONALITY... The CU-units manifest EE-units, and impress 'themselves' in the forms 'they' create, the CU-units manifest SOUND in its various-aspects, etc.

It all boils down to this: Any higher-science must seek out the 'consciousnesses' that lie behind all manifestations, period.



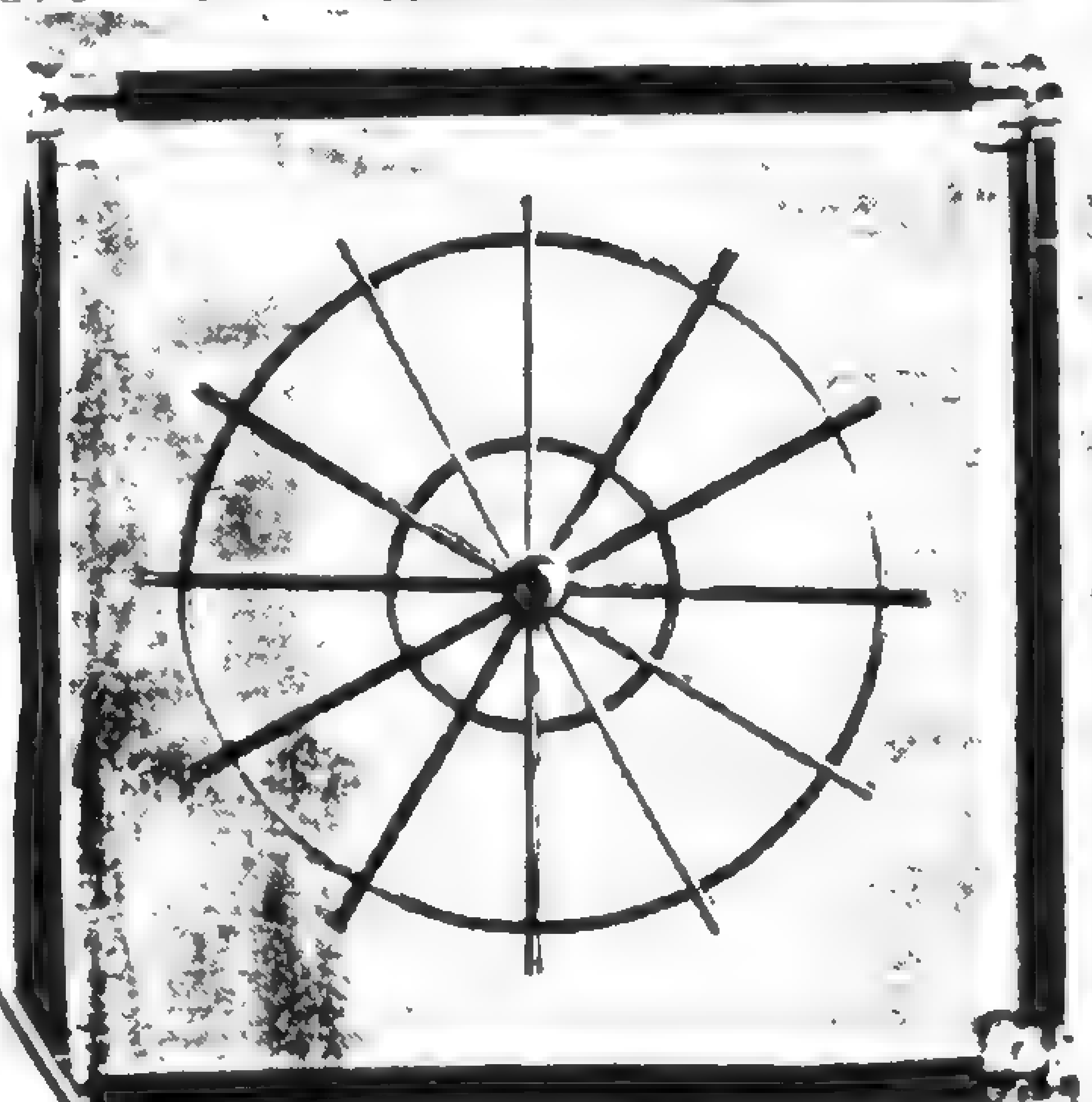
This is a hand-held crystal/shell device which is used for balancing energy centers of the body. It is excellent for activating the 3rd eye and accessing past life records and information from your ancient pasts and futures. Its compact design may be deceiving. It uses crystals to accelerate the naturally occurring vortex energies of the shell to extremely high rates of energy. A real blast from the past!

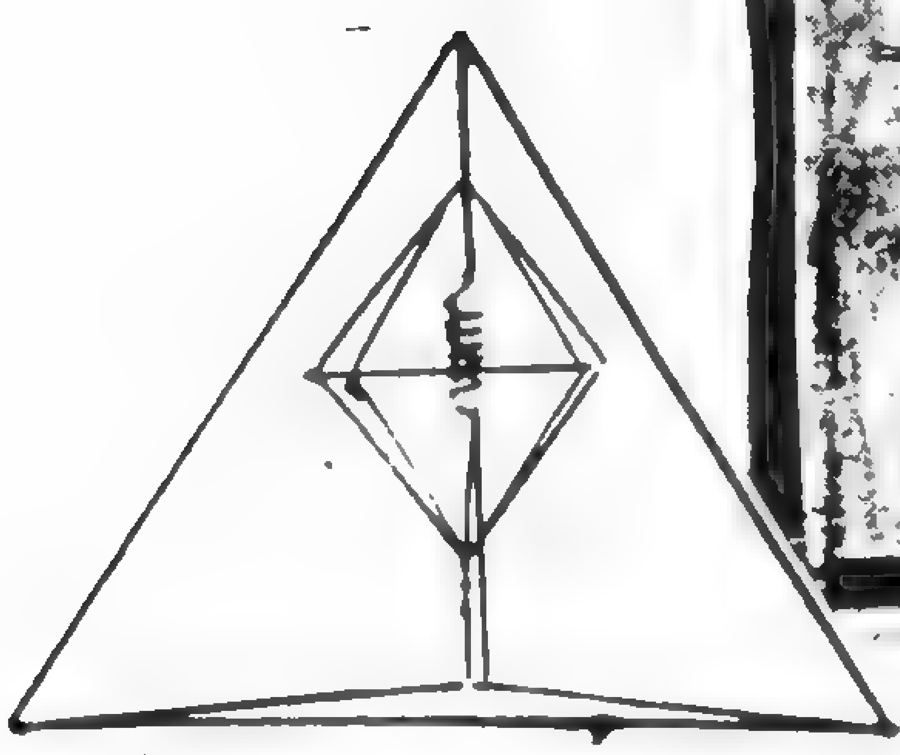


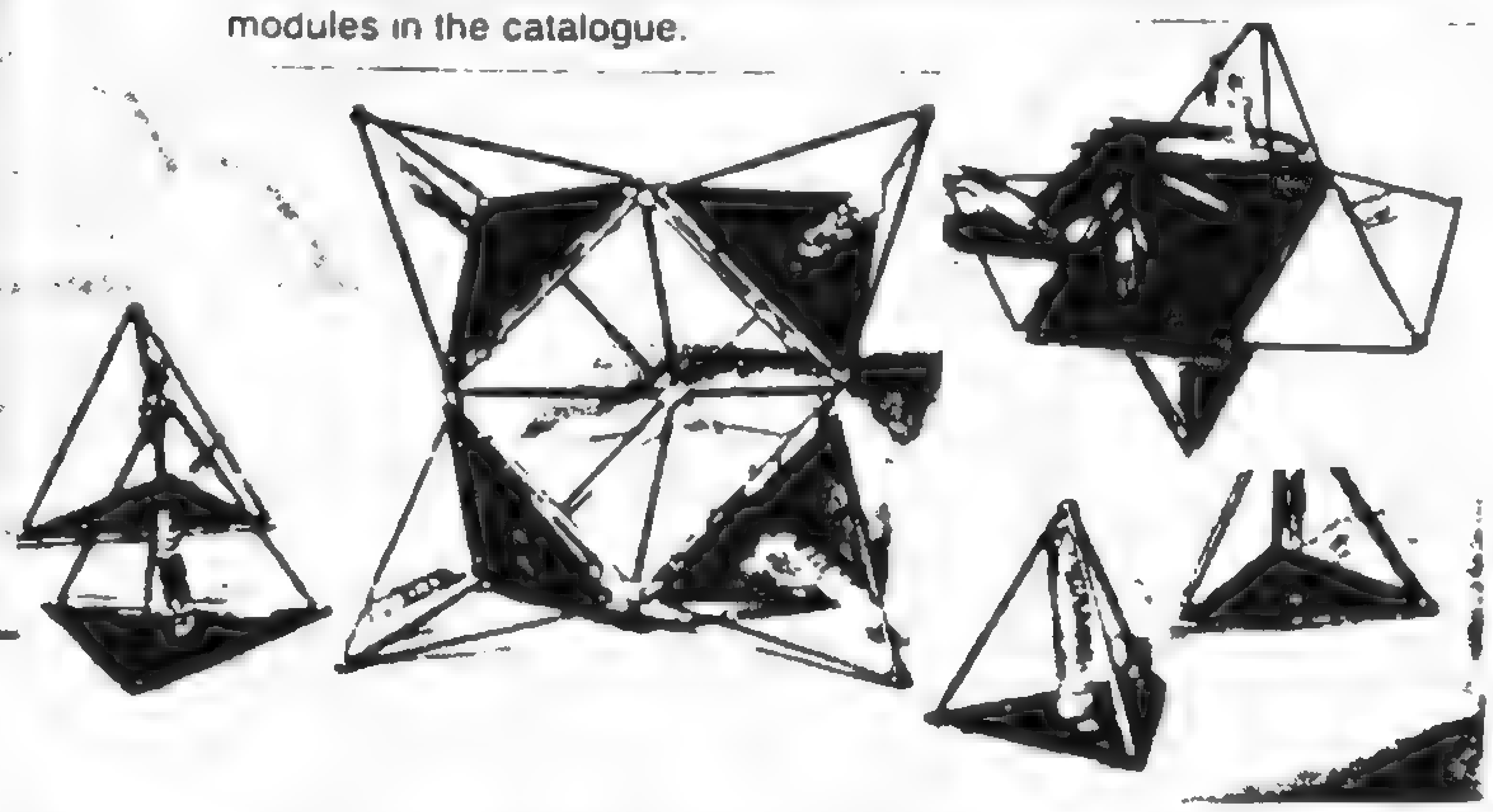
This device radiates orgone energy and enhances the astral bodies of those present. Its effectiveness can be enhanced by using pyramids and other geometric forms on top of the plate. The copper core is removable and can be used as a psychotronic witness well. This well may be used for transmitting colors, ideas and desired outcomes to persons, places and things present or at distant locations. Inter-dimensional communication is possible using this equipment. Color gel wheels are included for the enhancement of the signals and the tones created. This design is also effective as a decorative mandala for your enjoyment and meditation.

The *Solar-Fire Disc* capacitor emits a 7.5 Hertz rate every 22 seconds to a distance of 3.5 d (d = distance from center of plate to edge). The *Solar-Fire Disc* has a 21 foot field in all directions

This *Disc* is a module that can interact and be connected to other modules in the catalogue.







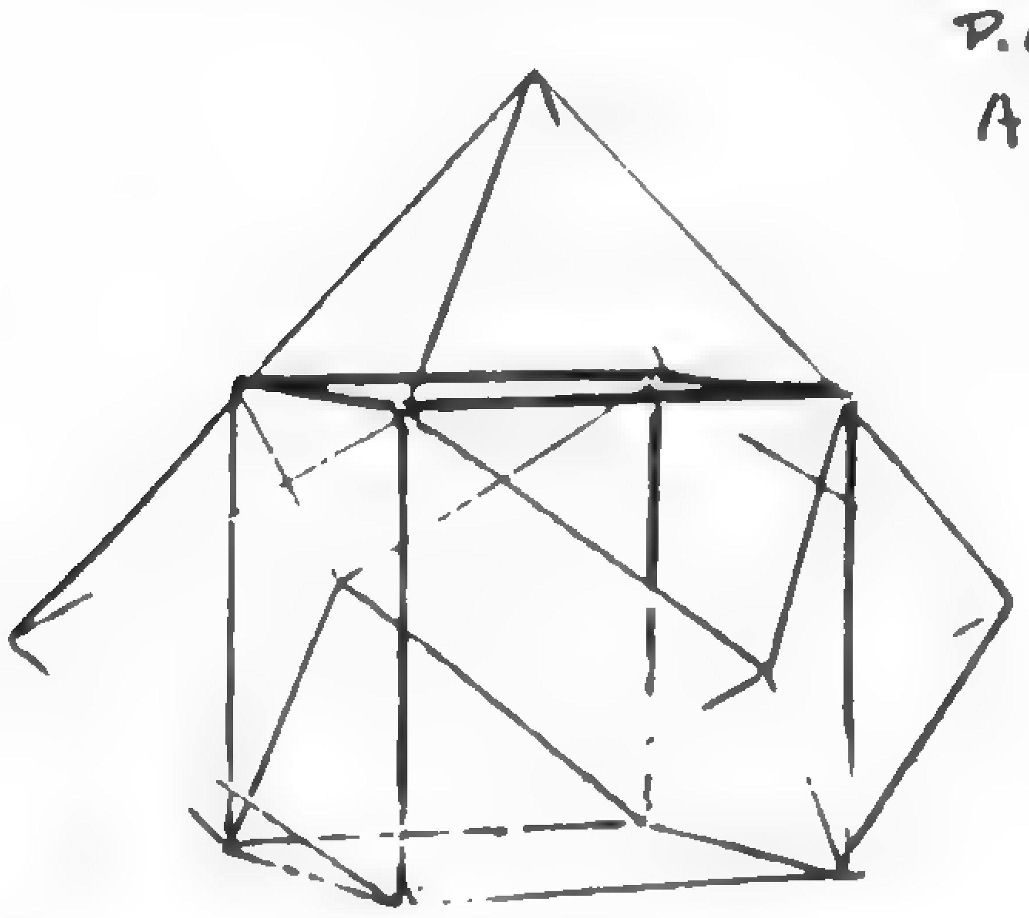
This copper tetrahedron stands approximately 12" with a 5" spinning octahedron suspended within it. Some of its assorted users are:

- A vortex aligner – many times there will exist places on the planet of unusual energies. This unit will aid in its rebalancing.
- The spinning of the central octagon creates an intense white light field.
- It can be used as a capstone for larger tetrahedrons.
- Meditating with this suspended form permits access to ultraterrestial and extraterrestial information levels
- Light body activation is accelerated for inter-dimensional time traveling.

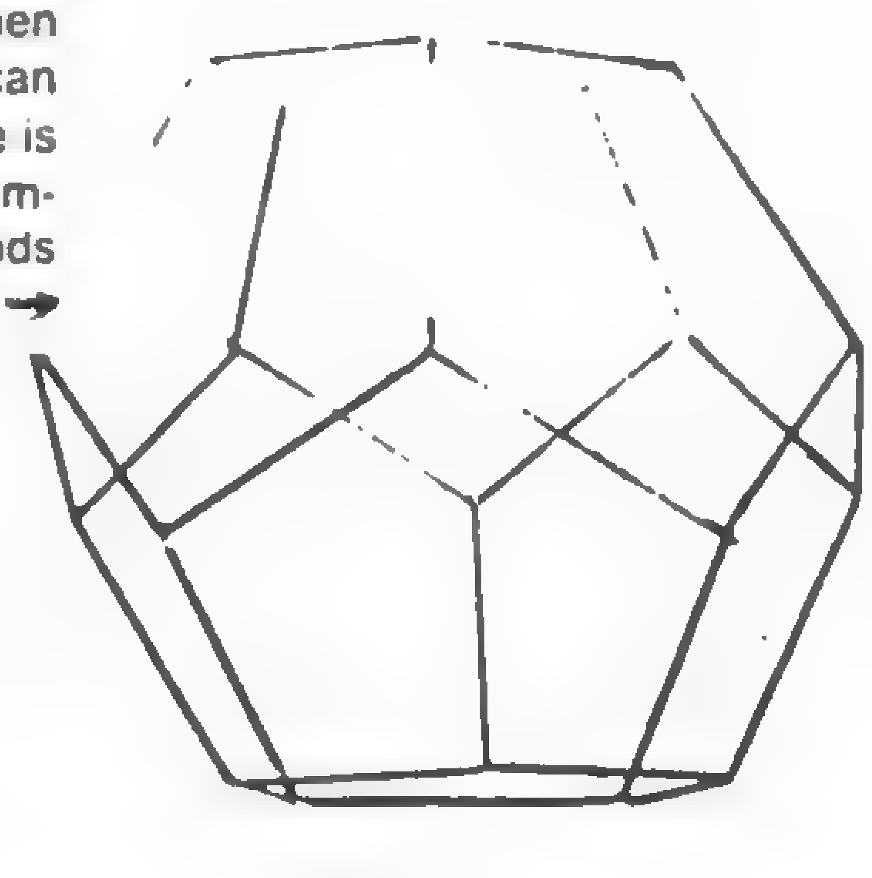
This device consists of an approximately 3" copper frame pyramid (capped with a 1/2" crystal pyramid) attached with bi-tetrahedrons on four sides. Included are four detachable tetrahedrons that plug in to create a synchronized vortex flow. Each tetra hedron is a miniature circuit board consisting of a copper wire wrapped crystal in its center. These detachable circuits can be placed on the body to enhance energy flow, energizing and balancing the chakras.

This time-view, Pre-Lemurian device can be used to communicate with Inner-Planet realms. The entire device can be suspended by string from the ceiling and sat under. It is advisable to have the device lightly touching the crown. Begin relaxation and meditative techniques of your choice and enjoy the *CLF Receiver* for a 20 – 30 minute session.

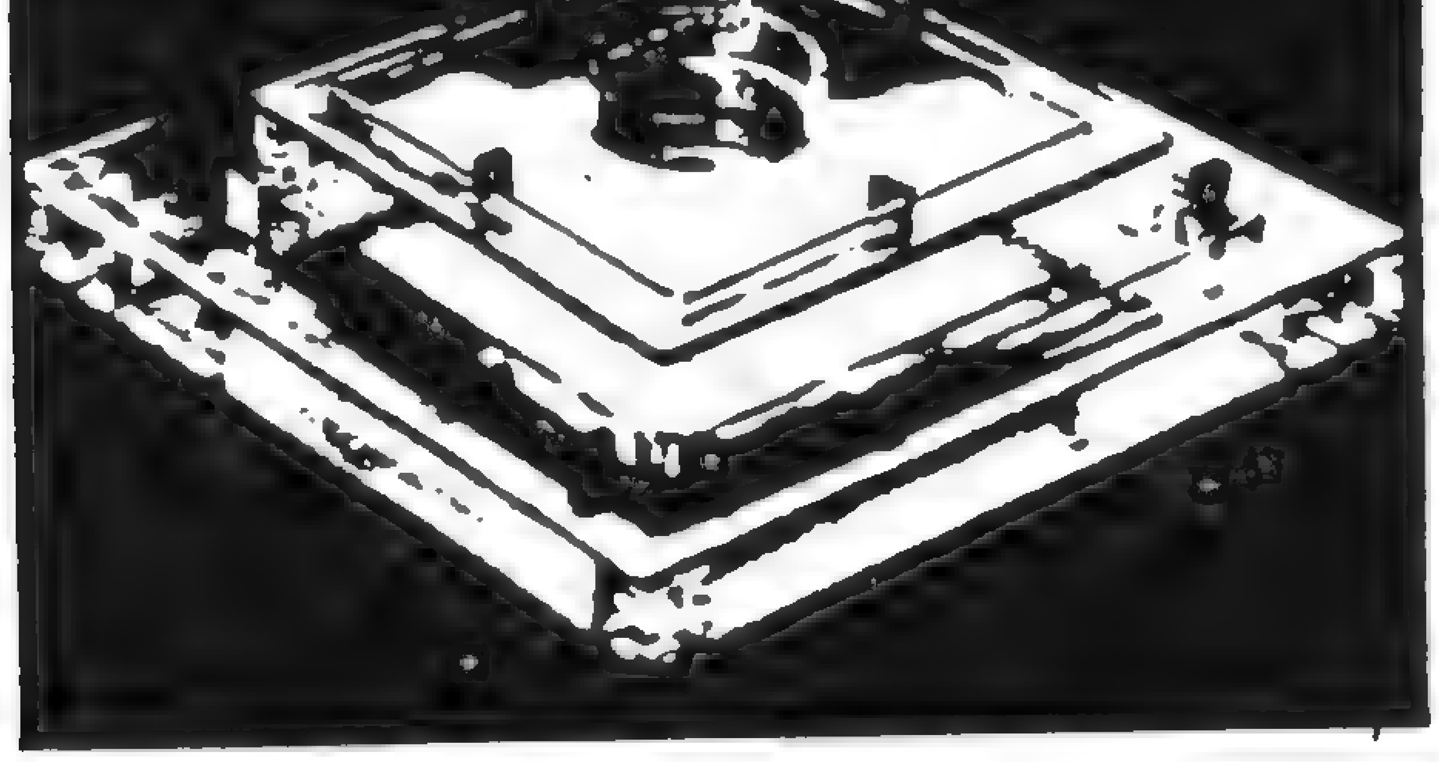
METRATRONICS UN
P.O. BOX 1131
ALIEF, TEXAS 90
— 77411 —



This geometric form, constructed of copper, is a resonant model of the planet earth. From out in space, the planet earth appears as a do-decahedron. When properly constructed with a refined metal, such as copper, this platonic solid can be used as a physical-energy representation of Mother Earth. This device is excellent for meditation and can deeply enhance your communication and communion with the Spirit of the Earth. Use of this device with crystals and pyramids can magnify results.



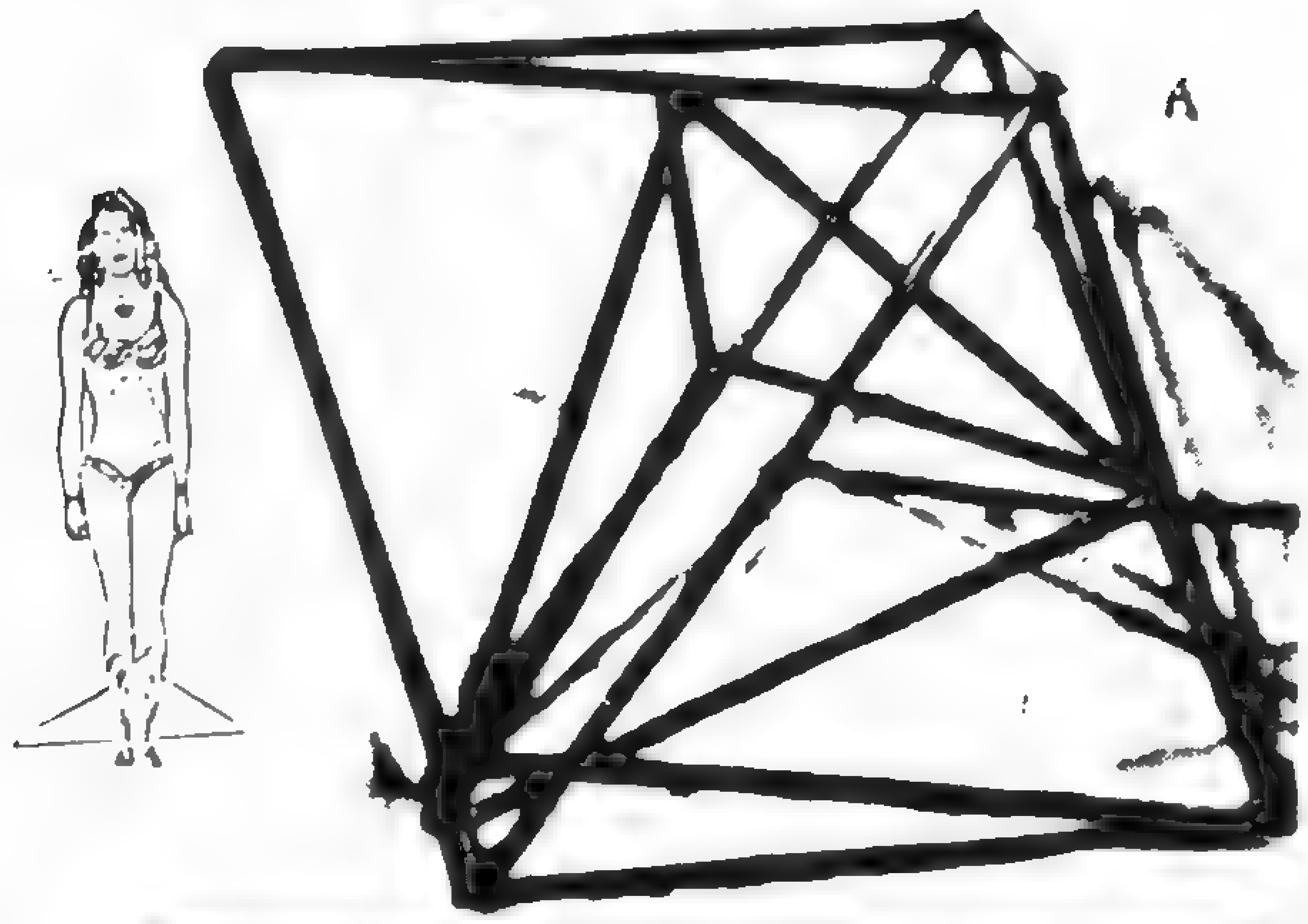
This device is constructed of five 6" base pyramids attached at each base. If placed on the top of a standard room size pyramid, it allows a totally efficient flow of the energy through the system. If hung by a string from the ceiling, it breaks up poor energy circulation and places the energy flow back to a normal balanced rate in the room.



This is an energy device which generates its own vibrant orgone energy field without the use of crystals or pyramids. Standing alone, it generates a stabilizing field of energy to approximately 300 feet in all directions. The beauty of each unit facilitates its use in the home or office setting either hung on a wall or sitting on a piece of furniture.

The *Mayamid Alphahedron/Omegahedron* can be used to charge, cleanse or detoxify by placing the item or a representation of the item in the central chamber. Using different gemstones will enhance and vary the quality of the energy transmissions.

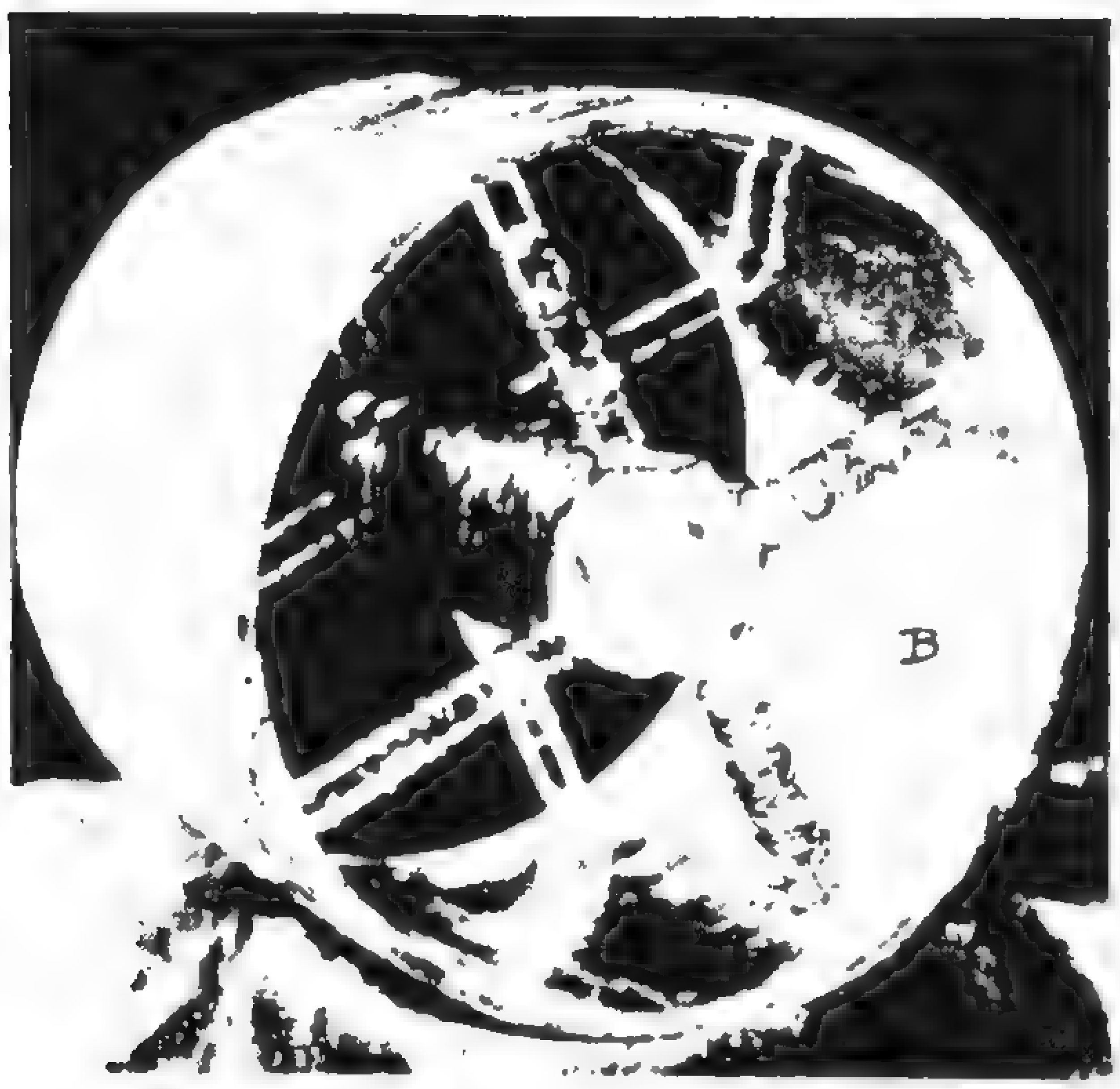
There are many used for this form so let your creativity by your guide.
This device comes in different styles:
*Clear Acrylic – smokey base – *Alphahedron*
*Black (high luster) Acrylic – *Omegahedron*



A

A This is an octahedron with an irregular shaped tetrahedron inside it. It is another "*Master Module*" which can be used alone or in conjunction with the *Star Pod*. When used alone, the *Theta* creates a variant wave pattern, slightly different from its Beta counter-part. The slight wave pattern enables the *Theta* unit to act as a cell rejuvenator. Its energy helps to vitalize the cell structure from the atomic/molecular level.

When the *Theta Master Module* is connected to the *Star Pod*, it amplifies the energies of the *Theta Master Module* by a factor of 10,000 – a synergistic effect achieved when a *Master Module* and *Star Pod* are inter-connected.



B

B This device is a miniature engine module which can plug into the *Solar Fire Disc Plate* and the *Starfire Module*. Connected to the *Solar Fire Disc Plate* this device acts as an amplifier'booster for its field. Essentially it is a crystal power pump. It is a multi-stage amplifier which is used to boost an existing field of energy. Its core energy is crystal.

C This is an energy device which generates its own vibrant orgone energy field without the use of crystals or pyramids. Standing alone, it generates a stabilizing field of energy to approximately 300 feet in all directions. The beauty of each unit facilitates its use in the home or office setting either hung on a wall or sitting on a piece of furniture.

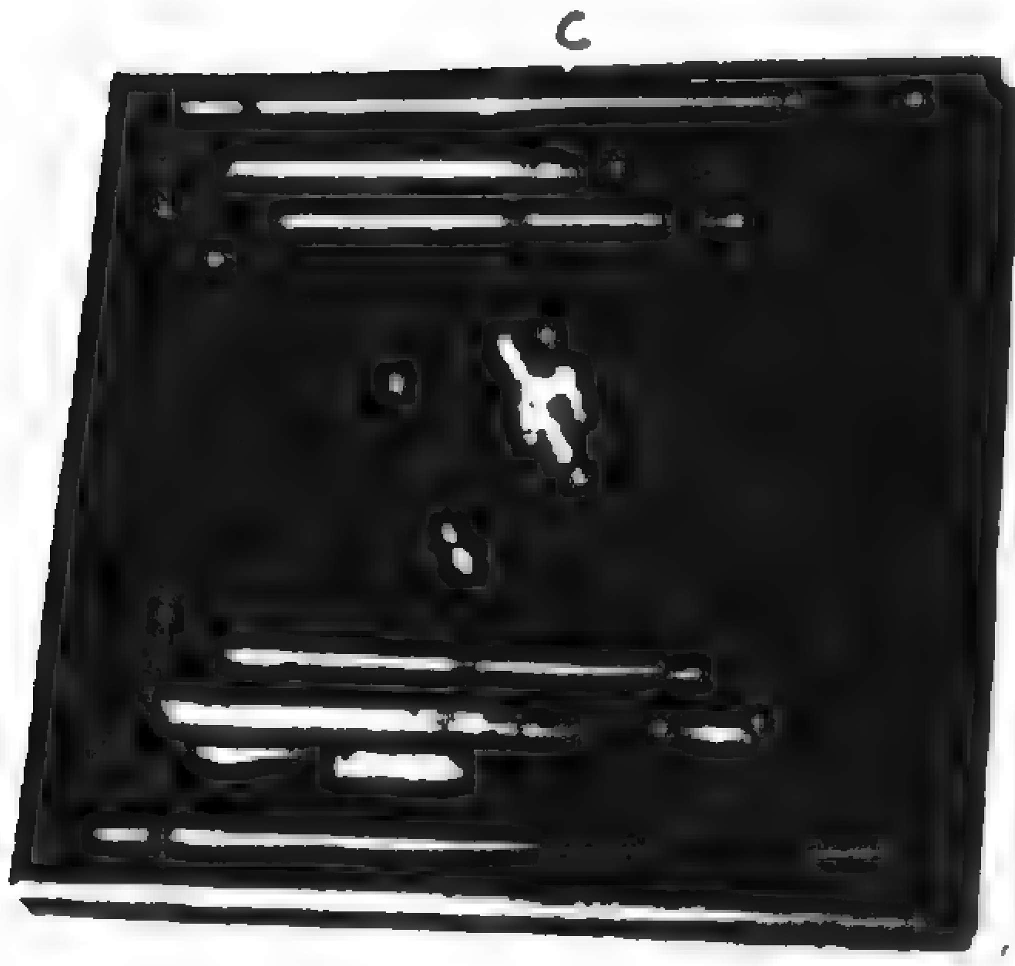
The *Mayamid Alphahedron/Omegahedron* can be used to charge, cleanse or detoxify by placing the item or a representation of the item in the central chamber. Using different gemstones will enhance and vary the quality of the energy transmissions.

There are many used for this form so let your creativity by your guide.
This device comes in different styles:
*Clear Acrylic – smokey base – *Alphahedron*
*Black (high luster) Acrylic – *Omegahedron*

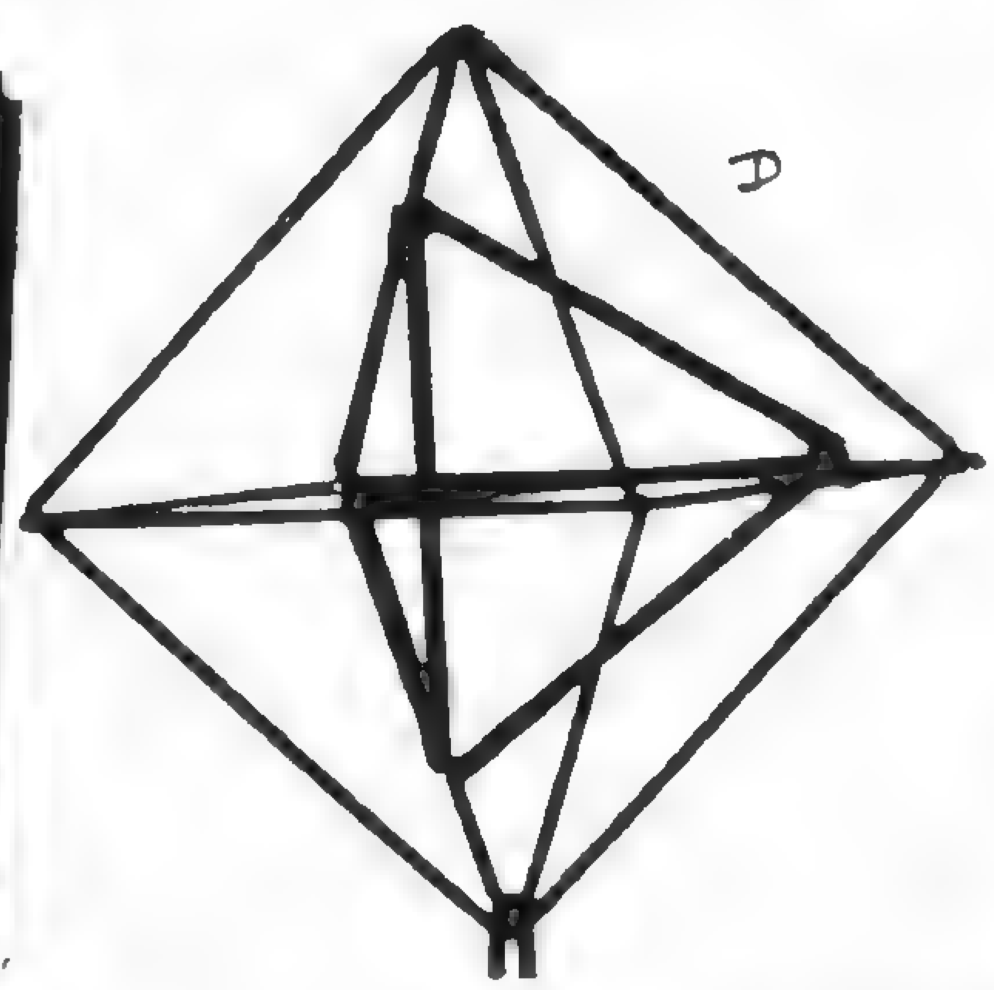
D This is an octahedron with a regular tetrahedron inside it. It is what is referred to as a Master Module and can be used alone or in conjunction with the Star Pod. When used alone, the Beta acts as an intensifier/balancer of the fire energy of the tetrahedron/pyramid complex. It is a signal enhancer to cell DNA to activate DNA codes in the production of third generation DNA patterns.

When the *Beta Master Module* is connected to the *Star Pod*, it engages the *Star Pod's* crystal harmonics with those of the module, creating an envelope of healthful and stimulating energy. A master module, used with the *Star Pod* amplifies the effect of the *Master Module* by a factor of 10,000.

C



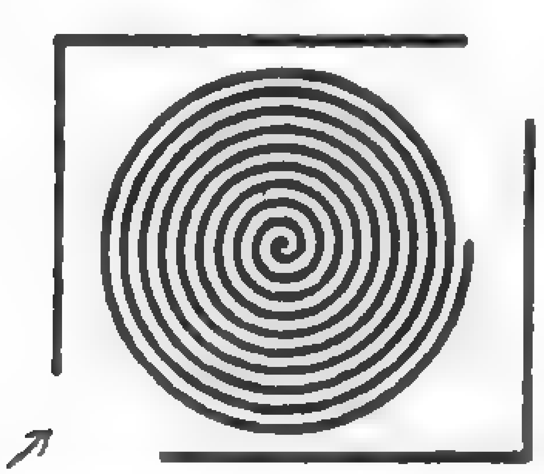
D



This device is literally a crystal engine or a mult-phase generator. It is composed of 108 crystal points arranged in specific geometric patterns which energize two Brazilian crystal points. inside the construction of the *Star Fire Module* is a chamber in which is placed a crystal, stone or symbol to key the module into operation. (It's and on/off switch.)

When in operation, the *Star Fire Module* creates an intense field of pure crystalline ultra-white LIGHT. Its field strength energizes from the sub-molecular level on out. This form creates an excellent energy field to induce creative thought processes and communication with the Angelic and Divine Creative *Forces* of *The All That Is*.

As an "art form", one half of the *Star Fire Module* can be placed on a light box with a color gel to create a soft, tranquil celestial mood.



ONE FORCE

Geocosmic Optical Designs
For
Subtle Energy Enhancement

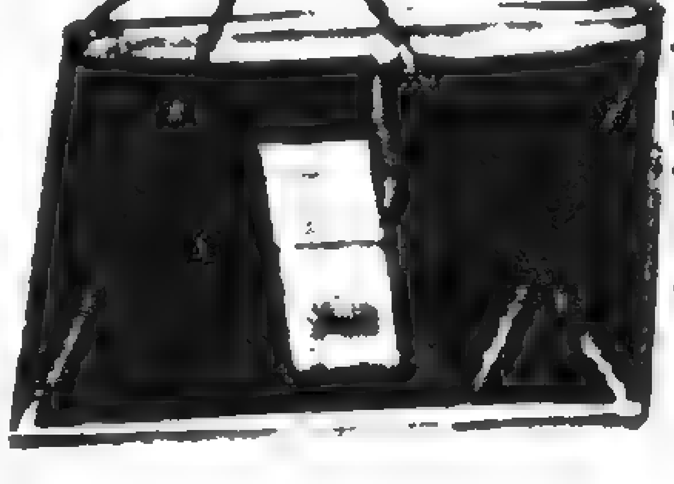
PO Box 1131
ALIEF, TX
77411

A new and yet ancient way for you to interact with the unlimited forces of creation and beyond. Once you begin to work with these forms, the subtle yet awesome forces at play within each design enhance quantum leaps in consciousness within your Self and your worlds.

There is but ONE FORCE in all worlds. The manifestations of this FORCE in the seen world (Electricity, Magnetism, etc.) are but dim reflections of THE ONE FORCE that exists in ALL THAT IS. We do acknowledge that what we offer in this catalogue are inspirations from THE FORCE.

We have been shown that the items available can be viewed as art, sacred sciences such as geometry, symbology, numerology, radionics, and mathematical relationships etc. All is true . . . for we have created a vessel, in each Geo-Cosmic Design, for this FORCE to flow through and be filled there-in.

How you interact with THE FORCE in each design is left to your creative desire. Some examples for use will be suggested . . . but . . . this is in no way excludes other uses. THE FORCE interacts with you in unlimited ways to achieve expansion of consciousness into ITS FLOW. This FORCE is subtle in ITS NATURE but awesome in ITS capabilities.



This device radiates a full color spectrum into any room it is placed. In other words, by placing it in a room, it will balance out any imbalance in energy via the color spectrum. By placing the name, witness, and specific requests on the king chamber platform . . . the balancing needed by this individual to manifest wholeness in areas of their life will be transmitted.

Nature's Geometry

"Of all people today, I think scientists have the deepest faith in the unseen world. The greater the scientist, the deeper his faith . . . It's a special brand of faith. You might say that the scientist sees God as a mathematician."

—Alan Lightman

The Brotherhoods have provided a great deal of information to assist us in understanding ourselves and our world. The cosmology presented in **The Ultimate Frontier** offers a working model of the nature and structure of the physical universe. A "scientist-philosopher" is not content to merely have "information." He seeks to understand the mathematics and physics underlying the nature of existence. He seeks to turn information into knowledge that serves Mankind. This search has lead to exploring the role of geometry in nature and science—nature's geometry.

Look around your environment for a moment and identify the various geometric shapes you see. Everything is a rectangle, circle, triangle, or square, right? Probably you answered yes, even if you saw some shapes that you couldn't quite define as "geometric." Our current way of thinking about geometry has been shaped by our culture's emphasis on the flat geometry of Euclid, which was invented some 2000 years ago. Even though Euclid's geometry is still regularly taught in schools today, as far back as 200 years ago mathematicians began to realize that all shapes did not fit into the geometries of the Greeks. Dealing with surveying and navigating on a globe led them to a "new," non-Euclidean geometry of curved surfaces. An example of non-Euclidean geometry would be triangles drawn on convex or concave surfaces. These would not look like triangles drawn on a flat sheet of paper! Non-Euclidean geometry eventually revolutionized mathematics and physics and led to Einstein's ideas on gravity and curved space-time, but still there were shapes, particularly in nature, that did not fit into either type of geometry.

The late Twentieth Century has seen a new form of geometry arise which gives science a better model for the real world. A mathematician named Benoit Mandelbrot has created a quiet revolution in science with a new geometry that describes many things in nature, such as trees, mud cracks, clouds, mountain ranges, coastlines, and water turbulence, things that were left out or avoided in the mathematical picture of the older geometries. The shapes of this new geometry are called "fractals." Two important aspects of the fractal concept are self-similarity and dimension.

"For example, a smaller part of a cloud is similar to the whole cloud; the veining on a leaf is similar to a twig or a tree branch which are all similar . . . "

Self-similarity means that a part of something is similar in structure to its larger whole. For example, a smaller part of a cloud is similar to the whole cloud; the veining on a leaf is similar to a twig or a tree branch which are all similar to the tree, and a creek with its feeding ravines and ditches is similar to the river with its tributaries. You can also identify many of these self-similar fractal shapes in your body, such as in the circulatory system and lungs.

Dimension is another important aspect of fractals, but with a difference. We're used to thinking of a flat object, say a leaf, as having two dimensions, length and width; and we find it easy to perceive a third dimension, depth, in a solid object such as a tree trunk. How many dimensions does the single object called tree have, though? It wouldn't work as a solid, three-dimensional object because of the tree's need for air and sunlight to reach the leaves. For the same reason, not even the leaves of the tree would work as purely two-dimensional objects; they have to have space around them in order to survive. If not two, but not three, then how many dimensions does a tree have? Fractal geometry solves the dilemma by thinking of dimension as able to be broken up into fractional parts (hence the name fractal). It considers a tree's fractal dimensions as being more than two but less than three. Fractals have truly created a new way of looking at the world, and made geometry more than a textbook subject. Instead, geometry has become a living, growing science in which the new builds on the past etc

Continued on page 8.

Dear Fry.

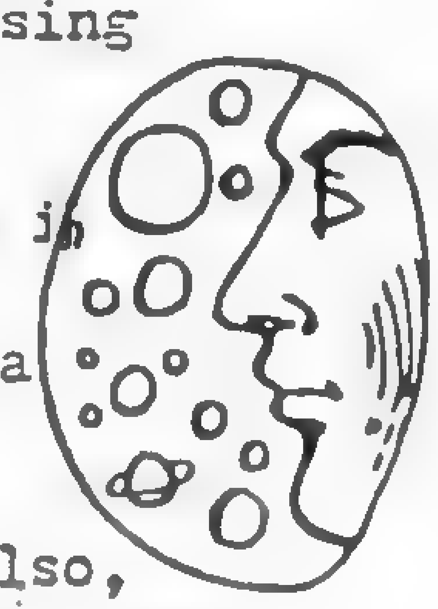
FROM: BRANDES LETTERS COLLECTION ($15.00 Fry's)

Received your letter today ... thanks again for the H-book orders. 5-23-80

Indeed! We have many X-minds in alternate-realities...the multidimensional-PSYCHE uses PSI-mind energy-gestalts to manifest reality-frameworks, etc. On a smaller-scale, the desire-mind, feeling-mind, and body-mind interlock-together forming an energy-gestalt operating within the physical-universe, however, the SPACIOUS-MIND OPERATES outside the physical-universe and has its own mental-continuum 'inner-laws' ...

Tapping-into the spacious-mind brings about numerous new-kinds of CREATIVITIES OF ANOTHER ORDER OF ENERGIES...using the spacious-mind,matter-teleporting of the physical-body on a planet or other galaxies becomes a practicality—changing the PLANCK-constant to alternate quantum-values(frequencies that overlap in a Fourier-sense) induces a correspondin change of the neuronal-synaptic -pulsings, and therefore altering the atomic-polarity of the physical-body to alternate quantum-states of ACTION...(actions)

Just re-read a second time STAR WARDS by Richard Miller, published by SOLAR CROSS ~~P.O. Box 215, Campbell~~, CA 95008—retails for $25.00 including postage. A Real BOMBSHELL, Star Wards, much info on MIND-TELEPORTING the physical-body anywhere on this planet or distant star-systems, and much rare-info on 'machine-teleporting using HEISENBERG'S QUANTUM-PLANCK EQUATION for basic-teleporting principles—data on TELEPATHY, PSI, etc. © Fry's



The star-people use hyperspace-warps of I,II,..., XI times the speed of light in quantum-jumps changing the values of planck's-constant h in quantum-jumps, etc.

Great changes coming between 1980-2000 A.D. before the new-Earth arrives via a change in its atomic-vibes to a new 'higher-level'...

In the Calculus of Organizers booklet published in 1961, last-page, I stated also, the importance of the PLANCK-CONSTANT in changing the ATOMICITY of MATTER via MIND, etc.

Presently working on absorbing some new ideas in higher number theory using a basic university text current to date—however, the triune-primes transcends ALL number theory 'concepts' to date, going far, FAR beyond Gaussian Congruence-theory number-concepts. Soon some new-developments on algebraic-aspects of the triune-primes and some refinement of deeper prime-number concepts...

A certain continued-fraction development suggests strongly the validity of the PROBABILITY-PACKET HEISENBERG Teleporting-equation(Tensor-aspect readout)...

Perhaps soon the H-book hyperspace-concepts required to develop-out visual-aspects of some tensor-concepts applied to TELEPORTING... The H-book is for the new-EARTH-age commencing in 2000 A.D.... *H MATH BOOK $25.00*

Some interesting 'results' have showed up in NUMBER THEORY via the PRIMES. I worked-out my own personal-version using the CHINESE REMAINDER THEOREM, modified somewhat, appertaining to an infinite-set of number QUANTUM-values; in other words, X-systems exist having their own unique-type of PLANCK-CONSTANT that changes 'value' from 'system' to 'system'.

Using the SPACIOUS-MIND one can travel the inner-universes via alterations of the Planck-Constant to any 'value' desired—in practical-terms, this means changing the neuron-speeds of the neurone-cell 'firings' at the synapses—which seems to be at the tensor-centers of the physical-brain.

Using the SPACIOUS-MIND, it would seem as-if 'technology' were a primitive-science developed so far, for the non-physical-mind has no need for any kind of 'machine-technology'

It seems that the Suerni came 'closest' in developing a rudimentary-science of the 'mental-realm' based on laws from the mental-continuum—the Atlanteans were still 'stuck' at a machine-level of technology and could progress no further due to their SYMBOL-BELIEF-SYSTEMS 'pivoted-around(focused) 'machine-symbols' — a symbol-framework 'codified' around a machine-technology.

For me, it's defintely a 'cop-out' from a machine-technology, for using the non-physical-mind, U can create any 'environment' U want once the 'mechanics' appertaining to the mental-realm is discovered and APPLIED. Beyond the mental-realm lies the inner-multidimensional-core of 'being' of each of us, and having unlimited-powers of creativity —unpredictability seems to be the 'basis' behind 'all' reality-manifestations.

Every manifested thought-form that has 'evolved' has its archtype-source within the inner-realms, the source-bed of ALL OUTER-MANIFESTATIONS.

I am spending some additional time 'studying' some number-theory 'concepts' obtained from an up to date number theory textbook written by Ralph G. Archibald. The text is 'loaded' with theorems and proofs(which I like very much...) and problem-sections without answers to make one THINK out-the-solutions, etc. However, I gained much new-math-awareness of another ORDER OF IDEAS lying outside 'continuum-number-concepts'. The text implies strongly the heavy-reliance on the PRIMES(prime-numbers) to solve number-theoretic-problems.

Complex-continued-fractions on a higher-order-level of application to quantum-mechanics solves matter-teleporting 'theory' and mechanics 100%.

However, the triune-prime arithmetic that I worked-out and developed goes far beyond the number-theory concepts known today, beyond the Gaussian-concepts of congruence-theory of numbers per se...

The prime-arithmetic can be summed-up very succinctly as follows: ALL facts, all realities are CONDITIONAL; that is, the 'facts' or 'laws' of 1 reality-framework may be partially-true or valid in an alternate-reality-framework, or again toally non-valid in yet another alternate-reality-framework, what COUNTS is the INTERRELATIONSHIPS between the GESTALT-units that uniquely-determine GESTALT-FRAMEWORKS(ACTION-GESTALTS).

The integers of arithmetic could be compared to the infinite-number of MATHEMATICAL-STATES of X-KNOWLEDGES existing in a PROBABILITY-CONTINUUM waiting to become manifested.

TENSOR-CONCEPTS can be applied to the H-graphics to gain visual -insight REPRESENTATIONS of algebraic-concepts and geometrical-applications, etc. I have about 50 additional-pages of advanced H-graphic material that is quite 'mind-blowing' when compared to the limited 3-D visualization-perceptic-processes—the H-graphics could become developed into an ART using a 'graphic-computer' to map-out numerous hidden-aspects of the H-material.

In some probable-futures which I envision, 4-D and 5-D computers are used as hyperspace-navigation 'tools'(traveling the different universes via quantum-warps)... A pun: How about SYMBOLIC-SPACESHIPS of the FUTURE?

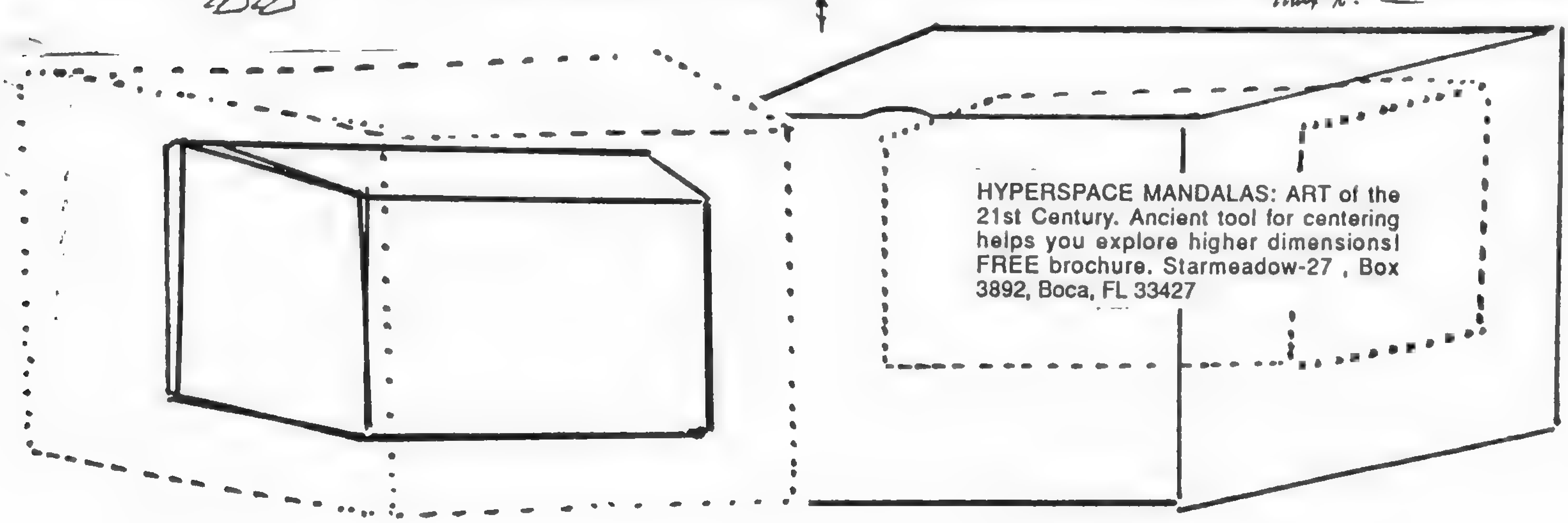
FROM: BRANDES LETTERS $15.00

Crystals are used to generate what we call the five Platonic solids. The most efficient of these is the cube. From this form generates all Light forms. When a cube is placed on its corner, it can be seen that it is made of two interlocking tetrahedrons.

All planetary bodies are composed of a central cubic core of crystalline energy. Stars contain the expansion of the crystal energy cube . . . the *Doublet Tetrahedron (DT)*. It is a double inverted fire-energy matrix, capable of generating the matter and anti-matter life forces to produce infinitesimal illumination.

As we get closer to the Star Treck technology depicted on our boob toobs, the entrenched power systems & energy cartels continue to struggle to hold such technology at bay. Already,the 'elite power groups have technology far beyond what we see at NASA. The time -space craft are only around the corner and the British Science fiction TV program Dr Who gives a pretty clear view of the possibilities. The TARDIS is a 5th dimensional teleportation vihicle contained in a 3 dimesional object. A London police call box. The machine is actually larger on the inside than the outside. Once inside the smaller box or object,the 5th dimensional distortion would produce math dimensions greater than the object in which the field is contained. You might note that the 3-D objects normal time & space measurements would not exist while the 5th dimensional temporal distortion was taking place. The field could be made to contain great amounts of space

Thought: — Bob Boyle



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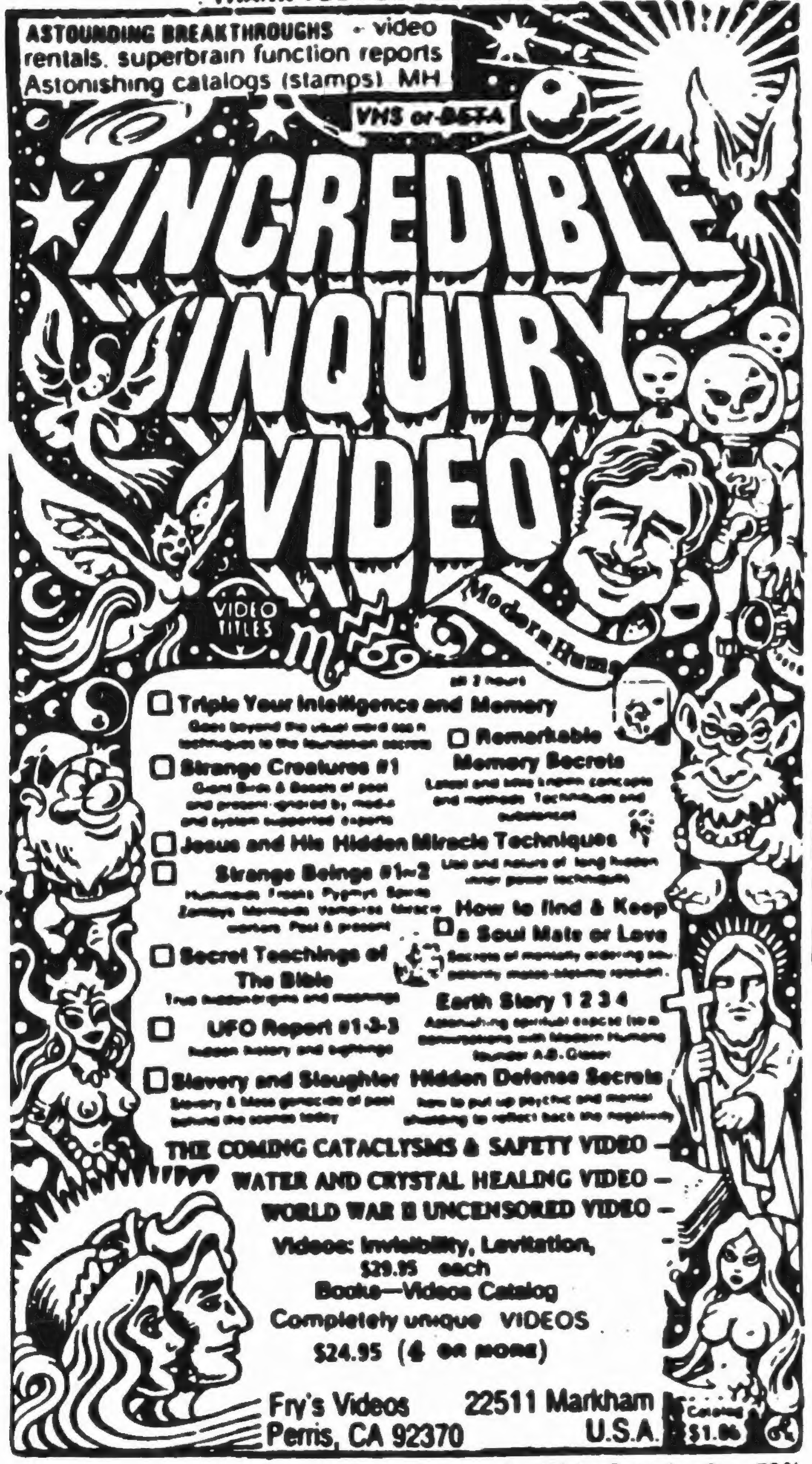

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WORLD WAR II UNCENSORED VIDEO — Learn how our popular leaders and war heroes were manipulated by the power structures who profited from and encouraged the bloodshed. We still never hear the truth of any recent wars from books or the media. See it all in great classic action film footage of the period. This may turn your entire views of the war around. —

WATER AND CRYSTAL HEALING VIDEO — Many little known systems can be used to purify and charge up water with energy. Most purification systems spoil the health benefits of water. You may as well find out how mother nature charges water for free. Willard Water, C Cells and many unusual water treating methods are covered. Plus a section on using Crystals in healing. 2 hours

THE COMING CATACLYSMS & SAFETY VIDEO — Present areas over much of our nation will be under water in the early 90's. If you remain, you and all your expensive toys will disappear. Learn what is going to happen and where to go for safety while you still can. Present Gulf streams are way off course and this will alter weather patterns well beyond even the chaotic patterns of the present. Disaster and quake zones all across the country are never mentioned by so called experts and you had better learn the truth. Computers, past record cycles, Indian prochesies, hynotic induced time travelers, psychics and others all point to disaster in the near future, and this two hour Video will boil $50 worth of books and searching time into weeks . . . into a very entertaining Video surprise. Lots of Px, maps & evidence. —

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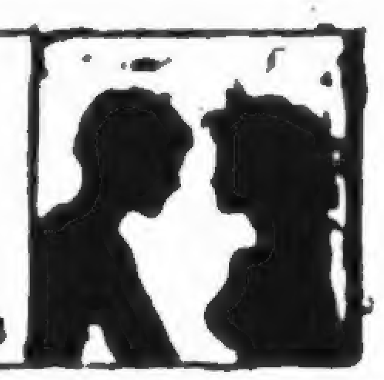

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--- FRY'S ---
879 Park Ave.
Perris, CA 92370

Greetings in the "Light" of Our Infinite One. I am Lalur. It is my intent to speak on the subject of healing. This field of endeavor, my brothers, is one which consumed many, many periods of time in our evolution. We, as you have been informed, have very little need in our present time for a medical profession, except in our exploratory craft when new and alien life forms are encountered. How this came about, how we were able to overcome that known as (1) disease, and (2) the decay and physical aging of the human body, I would like to discuss. First, if I may be permitted, I would lay some preparatory ground work.

In our early encounters with that known as disease, one parameter always was recorded on our charts in measuring the emanations about the physical body. Disease, in the many various forms which it can take, creates primarily a state of electro-chemical unbalance in the primary life force of an entity. By unbalance, I would use as an example one of your primary diseases, that known to you as cancer. In cancer, you have noted what may be classically called "a cell gone wild." All of your medical world have attempted to explain the erratic behavior of cancerous cells. It is what you would call a disease common to your present state of evolution. It is a mutant disease, one which could not have had its awareness in the cosmic meaning prior to a certain stage of evolution occurring upon your planet. I shall not explore fully all of the background concerning the reasons for mutation of cells, nor of what happens to the electron and proton, particularly the mutated nuclei of a cellular atom. I will merely mention that the normal cell electron departs from its polar orbit on an erratic course, caused by the bombardment of what is known to your peoples as cosmic radiation. It, my brothers, requires high energy particles to knock an electron from its orbit. This electron searches for a new state of balance in an adjoining atom. When it joins this atom, a change occurs. Your bodies are living examples of atomic fission and fusion.

Now we know that disease is an unbalanced condition of an electro-chemical nature. How do we rectify this? How do we restore balance to a diseased organism? This is the problem facing your world. Before I attempt an answer, let us talk for a moment about age.

The classical theory, popular amongst your peoples, is that tissues lose their vitality in the sunset of that known as physical life, due to an electro-chemical reaction taking place in the cells -- popularly speaking, a slowing down of that dynamo known as a cellular atom. Classically speaking, a slowing down of orbital velocities. Naturally, the tissue involved withers and wrinkles. The very essence of life carried throughout your bodies by your circulatory system is constantly supplying fresh atomic fuel to be burned in your body furnace. Again, we have a problem, that of adjusting the balance of body chemistry and electronics.

Perhaps I have over-simplified to a great degree the ramifications involved in all forms of diseases, and very broadly speaking, the aging process. Let me now mention what can be done to (1) eradicate disease entirely from the human race, and (2) to arrest that known as the aging process at any given age desired.

You have a statement, a classic, I am informed, that says you will "fight fire with fire." In this case we will fight radiation damage to cellular atoms with radiation. You are aware, my brother of the medical profession, as well as several present in your midst, that experiments have been tried in your great schools of learning with the effects that various colored lights have upon living substance. It is true, vegetation is the primary study involved at your present level and time. Certain plants when exposed to radiation of a particular color frequency exhibit unusual growth characteristics. They mature earlier. Should they be a fruit bearing plant the fruit is greater in size, it is more palatable to your taste. In every way the vegetation responds to a frequency of color vibration.

All of you are aware that the range of the human vision spans only several octaves of what is called the light spectrum of frequency vibra- the entire magnetic spectrum. What lies beyond the range of human vision at its shortest wave length known as ultraviolet? A serious gap, a most serious gap exists between the far ultraviolet and that known as X-radiation. None of your physicists have examined this gap for the unique properties inherent in this part of the spectrum.

The far ultra-violet and beyond, though higher in frequency than that known as X-rays, gama, alpha and beta radiations which, you have noted, have destructive tendencies on cellular tissue, to still far shorter radiation unmeasurable at present in your technology. We find that radiation performs in its many different and strange ways. Let us start, for example, with that known as sound. Some sounds, you have noted, are pleasant to your sense organs while others cause disagreeable sensations. We cannot say that since some sounds create an unpleasant sensation that all sounds are unpleasant, nor can we say that all forms of radiation, such as X, and alpha, and gamma are bad for the reason that they cause destruction of living tissue. You will find in examining what is known as the electro-magnetic spectrum, periods of reoccurrence of radiations conducive and pleasant and stimulating to the human organism, and those that are the opposite. So it is logical to expect that beyond the shortest radiations known to your present technology there lies a certain range of beneficial radiation frequencies. Through our many, many periods of time exploring the effects of radiations upon the physical body, we have made several interesting notes.

Let me first make a statement. This your evening, you are all being exposed to radiation. Radiations from your local radio broadcasting stations are passing through your bodies at this moment with little or no effect. At the same instant, however, cosmic radiation passing through your planets ionosphere is wrecking havoc upon the electronic balance of your neuro cells. One with no affect, the other totally destructive.

The answer, my brothers, to your classical problem is that in order to abolish that known as disease, we restore balance to the physical body, electrically and chemically. We employ several different forms of radiation, entirely unknown to your present technology.

Now, for the benefit of the medical doctor in your midst, should there be questions more imminent in this one's mind, may I be of some service?

- - - - -

Doctor (visiting us on this occasion): We have found a rhythmic impulse persisting throughout life within the cranium. The normal cycle is from ten to fourteen per minute. We have found this increased in fevers, decreased in mental conditions, varying with other disease conditions. We do not know what it is. We think it is due to a fluctuant cycle in the cerebrospinal fluid which controls the metabolism of the brain and central nervous system and, therefore, the whole body. We feel it is, in a sense, the highest known element in the human body, possibly our link with our Creator. We would like to know, because of the significance of this fluctuant wave or cycle, because of its possible diagnostic and prognostic value, we would like to know how it can be recorded on an instrument acceptable to the scientific world, usable by any and all physicians.

FROM $25 'STAR WARDS'

FROM: BRANDES
LETTER #15

CREATIVITY

5-14-83

Today discovered some extraordinary new-insights in the prime-arithmetic that literally takes my breath away, so subtle in nature that I earlier missed it completely.
Finally got the transforming referencing-symbol-operator to work 100% . It appertains to the unlocking-locking process via the prime-coprime dissolutions of any order.
I am utterly astounded completely at the GESTALT-PROPERTIES of the cordella-primes , it is like nothing in the entire realm of present-day 'math'. LANGUAGE-structuring was an unsolved-problem for me for quite a while since I had to UNSTRUCTURE the thought-processes via the old-language into a new-language corresponding to the gestalt-prime-like processes, it was not easy due to the old-conditioning that kicks-off every so often and I have to catch myself. Here's my programme-key-in:

- Data-facts in one FRAME of REFERENCE may NOT apply at all in another FRAME of REFERENCE, and still in a 3rd FRAME of REFERENCE only PARTIALLY. This statement becomes tensor-oriented for conditionalities of the thought-process, NOT for old conditioned-states of thinking, etc.

Re-reading the compacted-form of Comp. IV, it had some excellent-ideas in it appertaining to the nature of the 'atom', i.e. its INVISIBLE-COUNTERPART blueprint that makes possible the VISIBLE manifestation of the 'atom'. But the KINDS of probable-atoms exist as stated in Comp. IV, and indeed, we have in a sense a CONTINUUM of CONSCIOUSNESSES' evolving-out along ALTERNATE-INFINITIES, etc.

Now what U call Amind, Bmind, and Cmind are NOT the OVERMIND, or the SPACIOUS-MIND overseeing the 3 body-minds. The body-consciousness(intellect) remains physical and this is ego-consciousness in part. The other two body-minds, i.e. the desire-mind and feeling-mind are non-physical and interlock with the body-mind consciousness. But the SPACIOUS-MIND directs the 3 body-minds in their functioning. The SPACIOUS-MIND exist OUTSIDE of this universe, more so, beyond SPACE and TIME, and the PSI-POWERS come into manifestation only by the use of the SPACIOUS-MIND; the 3 body-minds can't operate the PSI-powers unless the PSI-energies are channed to it from the SPACIOUS-MIND. The SPACIOUS-MIND operates the 3 body-minds via the TENSOR-CENTERS in the brain.
Present-day 'man' uses only the cortical-centers of the brain(20% of the brain-functioning), and a tensor-oriented 'man' would use the TENSOR-CENTERS(the other 80% of the brain functioning) in the reality creating-process. OUT-OF-BODY projections use the TENSOR-CENTERS of the brain...

SETH is right, "We create our own reality", and if we believe in limitations then we will meet them in our experience, but if we don't believe in limitations then we will not experience them—that simple it is.

Now the astounding part in what SETH said, "ALL THAT IS"(pure-energy) has

NO MASS".

More astounding of the astounding, SETH sez PURE-ENERGY can NOT be DIVIDED-UP like a pie into parts... . At the entity-level of creating, VERBAL-language does not exist, and creativity is direct without the intervention of SYMBOLS...

The inner-self(inner-ego) can handle infinite-probabilities within a twinkling of an eye, but the ox-cart outer-ego can only select 1 probability-outcome at atime.
The inner-self can be at oneness and apart at the same time, but to the outer-ego it gets lost at times going to extremes from apartness to oneness.
SETH sez the HUMAN PERSONALITY has no limitations unless it accepts limitations imposed upon it by the outer-ego.

Sincerely Yours,

George L. Brandes

P.S. 340-pages on the Prime-Arithmetic and still going strong with the new AIMS of creativity...

HOW CAN you squeeze troublesome inductance and reactance (resistance to changes in AC) out of a resistor? One way is to make a resistor in the shape of a Moebius loop—a century-old mathematical oddity that is based on a geometric surface having only one side and one edge.

Under ideal circumstances, a resistor should provide only resistance, a capacitor only capacitance and an inductor only inductance. Unfortunately, in high-frequency circuits (UHF and microwave) and especially in pulse applications such as radar, the design and operation of such circuits is greatly affected by the unwanted reactance inherent in these components. The higher the frequency, the more critical these parasitic values are.

A unique solution to one of these problems (making low-value resistors non-reactive) has been found by Richard L. Davis, an electronics engineer with the Sandia Laboratories in Albuquerque, N.M. Davis reasoned that if current passing through a resistor could be divided into two equal components whose electromagnetic fields cancel out, the reactance of the resistor would be small. How could such a resistor be made? The Davis solution was to add a simple Moebius twist to a ribbon- or wire-conductor resistor.

Kooky Loops. Perhaps the best way to visualize the construction (and operation) of a Moebius resistor is to make a couple of Moebius loops from long strips of paper that are about an inch wide. First make the basic loop by joining (with tape) the two ends of a single strip after you have given the strip a half twist. This loop has only one surface! Prove this by drawing a line along the full length of the strip, right back to your starting point (see lead photo). The line will cover both sides of the strip.

A Moebius resistor, however, must be constructed with two conductive ribbons, with or without a separating dielectric layer. So now make another Moebius loop, this time using two identical strips of paper, one on top of the other; again, give the strips a half twist before joining the opposing ends together. Label one of the splices *input*, the other *output*.

It may appear that there are still two separate loops—a pencil between the strips can be slid completely around the loop back to the starting point. Actually, there is just one loop. You'll see this when you attempt to separate them. The two paths that the input current will take to the output terminal can be traced once the loop is opened.

How It Works. The input pulse that's applied to one of the terminals divides into two equal components which travel in opposite directions. This happens because the impedances of the two paths to the output terminals are identical. Since one pulse loops to the right while the other heads left, they cannot interfere with each other. Then, when the pulses have traveled half way to the output—where DC resistance is one half the total value—the pulses are at equal potential and of opposite phase. By the time they reach the output, their potentials fall to zero.

The two terminals must be exactly opposite each other otherwise the resistor becomes inductive (the pulses wouldn't be 180° out of phase and residual magnetism would be present). While it is preferable to eliminate lead wires whenever possible (to avoid stray capacitance), a resistor that is slightly capacitive can be nulled into balance if you adjust the lengths of the leads.

Davis' first experimental resistor was made of aluminum-tape conductor placed on masking tape. The masking tape serves as the dielectric. It had a 0.022-ohm resistance and 0.003-μh residual reactance. The time constant (1.3 x 10^{-7}) was very low for such a small resistance. These values may seem ridiculously low to people who experiment at audio and lower RF frequencies, but as you get up into the spectrum such component values have tremendous effects on a circuit.

Tuned UHF circuits have resonant frequencies requiring almost invisible capacitors and coils, and the short wavelengths are too large for most any component. In fact, most radar circuits use resonant cavities rather than individual capacitors and coils. Cavities and waveguides act on electromagnetic fields, while resistors, capacitors and coils are designed primarily to control electrons flowing in wires. The former act like distributed constants, the latter are lumped constants. A Moebius resistor is a lumped-constant component.

Great Versatility. One striking feature of a Moebius resistor is that it does not couple electromagnetically to other metallic objects or to itself, even if the shape of the finished resistor is changed. There are only two requirements for this: the conductors must not touch physically and the spacing between the conducting layer must not be altered. This non-coupling characteristic makes it possible to wrap Moebius resistors around cards.

Moebius resistors are simple—and inexpensive to make. Problem is, unless you've got a use that works at frequencies from around 500 to 4000 mc, you won't find much use for them. Of course, if you're a mathematician you can always reach for a textbook on topology—just to find out what Mr. Moebius was really talking about.

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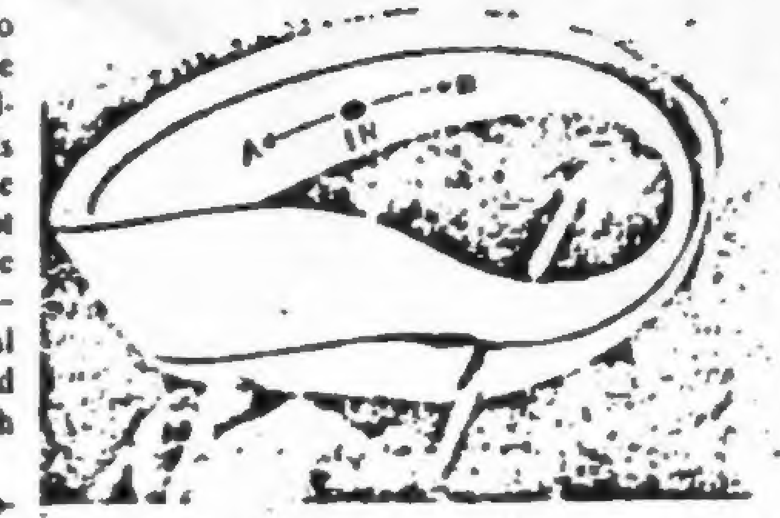
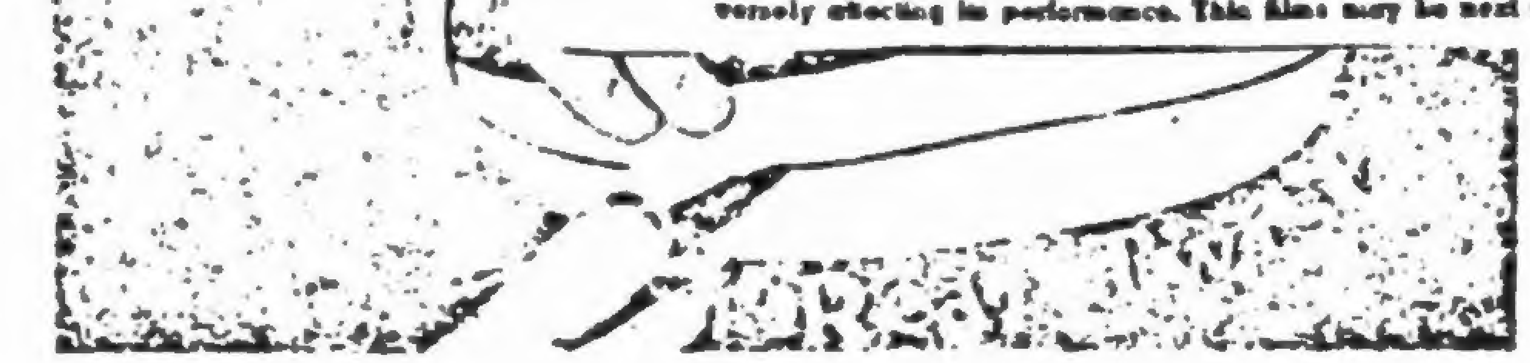


Two paper loops simulate Moebius resistor made of two conducting ribbons and separating dielectric. Input and output terminals must be opposite.

Moebius loop resistor has only one continuous conducting surface. Dielectric material separates opposing surfaces of the conductor. Leads are exactly opposite.

Richard L. Davis, inventor of Moebius resistor, shows how this component looks before it's bundled into a compact package.

Actual prototype of Moebius resistor is shown at right. The resistor strip can be wound into a compact bundle without adversely affecting its performance. This disc may be next design.



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